REU 2005 · Discrete Mathematics · Lecture 9

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1 Lecture 9

1.1 Monoids

Definition 1.1. Let M be a monoid, G a group and $f : M \to G$ an onto homomorphism we say that the pair (f, G) is a *universal group* of M and G is a *universal group quotient* of M if for any group H and any homomorphism $h : M \to H$, there exists a unique $g : G \to H$ such that $h = gf$. That is, the following diagram commutes,

$$\begin{tikzcd} M \arrow[rr, two heads, "f"] \arrow[dr, "h"'] & & G . \arrow[dl, dashed, "g"] \\ & H & \end{tikzcd}$$

Theorem 1.2. *Every monoid has a universal group quotient.*

Proof. We can define it by generators and relations: $G = \langle M \mid \text{ multiplication table of } M \ \rangle_{\text{group}}$. So the whole set M is the set of generators. □

For example, with the yo-yo $M = \{1, z, z^2, z^3, z^4, z^5, z^6, z^7 = z^4\}$, we get that $M = \langle z \mid z^4 = z^7\rangle$. Defining the universal group quotient as in the proof of the theorem, we get $G = \langle z \mid z^4 = z^7\rangle_{\text{group}} = \langle z \mid 1 = z^3\rangle = C_3$, the cyclic group of order 3.

Exercise 1.3. *Finish the proof of the theorem: show that the G thus defined is the universal group quotient.*

Observation 1.4. Additionally, every finite monoid has a unique **minimal** ideal.

Recall that $I \triangleleft M$ is an ideal iff $MIM = I$. "Minimal" means nonempty, but does not properly contain any nonempty ideal.

For the proof, recall the following

Claim 1.5. *If $I, J \triangleleft M$ for $I, J \neq 0$, then $I \cap J \triangleleft M$ is also nonempty.*

Proof. $IJ \subseteq I \cap J$ is a nonempty ideal. $\square$

Definition 1.6. This minimal nonempty ideal is called the core:

$$\operatorname{core}(M) := \bigcap_{I \triangleleft M, I \neq \emptyset} I$$

Theorem 1.7. *If M is a finite commutative monoid then core(M) is its universal **group**.*

Note that the finiteness of M is needed: for example $\mathbb{N}+k \triangleleft \mathbb{N}$ are ideals but $\cap_{k\in\mathbb{N}}(\mathbb{N}+k) = \emptyset$.

Proof. Now, let's consider: why is $\operatorname{core}(M)$ a group? That is, for $I := \operatorname{core}(M)$, why is $ax = b$ solvable in I? Well, $I \triangleleft M \Rightarrow aI \triangleleft M$ because $I = IM$ and $aI = (aI)M$, which shows that aI is a right ideal, which is the same as a two-sided ideal by commutativity. Now $aI = I$ by minimality, since $aI \subset I$ (I is itself an ideal). So we can solve $ax = b$ for any $a, b \in I$.

Next, we need a map $f : M \twoheadrightarrow I$ for $I = \operatorname{core}(M)$, where M is a finite commutative monoid. Take the identity $u \in I$ of the core I (which is a group). Then we define

$$\begin{aligned} f :& M \twoheadrightarrow I \\ f :& x \mapsto x \cdot u \end{aligned}$$

so that

$$xy \mapsto (xu)(yu) = (xy)(uu) = xy \cdot u,$$

as desired (this is indeed a homomorphism).

Claim 1.8. *This map is onto.*

Proof. For $y \in I$, $y \mapsto y \cdot u = y$ and u is the identity of I. because $y \in I$. $\square$

Next, we need to check universality:

$$\begin{tikzcd} M \arrow[rr, two heads, "f"] \arrow[dr, "h"'] & & G \arrow[dl, dashed, "\exists! g"] \\ & H & \end{tikzcd}$$

Well, for $y \in I$ we need to define $g(y)$ so that $(\forall x \in M)(h(x) = g(f(x)))$. To do this, let's just set $g(y) = h(y)$. Then we need $h(x) = g(f(x)) = g(x \cdot u) = h(x \cdot u) = h(x)h(u)$. This follows from the

Claim 1.9. $h(u) = id_H$.

Proof. $u^2 = u$, so $h(u^2) = h(u)^2 = h(u)$. Thus $h(u)$ is an idempotent, and since it lives in a group, $h(u)$ is the identity. $\square$

So we have proved the theorem.

$\square$

1.2 Sandpile Group

Recall that t_v for any vertex v in the configuration with one chip on t_v and none anywhere else.

Theorem 1.10. *M is the monoid generated by $\{t_i : i \in V_0\}$ subject to the relations*

$$\deg^+(v) \cdot t_v = \sum_{j \in V_0} a_{ij} t_j, \tag{1.1}$$

where a_{ij} is the multiplicity of the $i \to j$ arcs.

Proof. Let $\widetilde{M}$ be the monoid defined by (1.1). M satisfies (1.1). So $\widetilde{M} \to M$. On the other hand, using the relations (1.1) we can simulate chip-firing and so all elements of $\widetilde{M}$ can be written as

$$\sum_{i \in V_0} n_i t_i, \quad 0 \leq n_i \leq \deg^+(i).$$

Consequently $|\widetilde{M}| \leq |M|$ and so $\widetilde{M} = M$. □

Exercise 1.11. *The* sandpile group *is defined as an abelian group by the same relations as the sandpile monoid. HINT: Use the fact that the sandpile grop is the universal group quotient of the sandpile monoid.*

Recall that the **Laplacian** is the matrix

$$\begin{pmatrix} \deg^+(1) & & 0 \\ & \ddots & \\ 0 & & \deg^+(n) \end{pmatrix} - \begin{pmatrix} & & \\ & a_{ij} & \\ & & \end{pmatrix}$$

Note that the laplacian does not change if we add/remove loops.

The **reduced Laplacian** takes the Laplacian and deletes the row and column corresponding to the sink.

Then the relations (1.1), viewed as relations in an abelian group, become equivalent to

$$(\deg^+(i) - a_{ii}) t_i - \sum_{j, j \neq i} a_{ij} t_j = 0; \tag{1.2}$$

corresponding to the rows of the (reduced) Laplacian.

Corollary 1.12. *The sandpile group does not change if we add/delete loops.*

Recall that a **finitely generated abelian group** is something of the form $G = \langle \Sigma_{i=1}^k c_i g_i | c_i \in \mathbb{N} \rangle$. A **free abelian group on k generators** is of the form $\langle t_1, \ldots, t_k \, | \rangle \cong \mathbb{Z}^k$.

Define a homomorphism $f : \mathbb{Z}^k \to G$ by

$$\Sigma c_i t_i \mapsto \Sigma c_i g_i.$$

Exercise 1.13. *Proove that this is a well-defined map and is in fact a surjective homomorphism.*

Let $K = \mathrm{Ker}(f) = f^{-1}(0)$. We describe this by the sequence

$$K \longrightarrow \mathbb{Z}^k \overset{f}{\twoheadrightarrow} G\,.$$

Exercise 1.14. $G \cong \mathbb{Z}^k/K$ *(the set of cosets with natural addition).*

Let K be generated by $k_i = (\alpha_{i1}, \cdots, \alpha_{ik}) \in \mathbb{Z}^k$, $i = 1, \cdots, m$. We permit m to be infinite, although this turns out to be unnecessay. Now consider the $m \times k$ matrix $A =$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1k} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2k} \\ \vdots & & \ddots & \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mk} \end{pmatrix}$$

Then $G = \langle t_1, \dots, t_k \mid A \rangle$, where $K =$ all integer combinations of the rows of $A =$ the subgroup of $\mathbb{Z}^k$ generated by the rows of A.

Note that

Exercise 1.15. *Elementary row operations on A don't change the group K.*

Exercise 1.16. *If we perform an elementary* ***column*** *operation, then this does not change G; only the generators change. For example, $b_j \mapsto b_j + \lambda b_i$, replacing the generator t_i by $t_i - \lambda t_j$.*

Now, suppose A is in Smith Normal Form. (We note that this works even if A has infinitely many rows). Then we get

$$\begin{pmatrix} 1 & & & & & & & & 0 & 0 \\ & 1 & & & & & & & 0 & 0 \\ & & 2 & & & & & & 0 & 0 \\ & & & 6 & & & & & 0 & 0 \\ & & & & 6 & & & & 0 & 0 \\ & & & & & 18 & & & 0 & 0 \\ & & & & & & 0 & & 0 & 0 \\ & & & & & & & 0 & 0 & 0 \end{pmatrix},$$

so that $t_1 = 0 = t_2$, and $2t_3 = 0 = 6t_4 = 6t_5 = 18t_6$, with no equations involving t_7, t_8, t_9, or t_{10}, so we get the group $C_2 \times C_6 \times C_6 \times C_{18} \times \mathbb{Z}^4$. In this case, it is really easy to read off the structure of the group! And by the preceding, we can reduce the matrix A to this by row and column operations and thus read off the structure of the group G.

Thus we have proved the **FUNDAMENTAL THEOREM OF FINITELY GENERATED ABELIAN GROUPS**

Recall that the **rank** of an abelian group is the minimum number of generators. We have the

Exercise 1.17. *If the invariants of a finitely generated abelian group are $n_1 \mid n_2 \mid \ldots \mid n_r$ (i.e. the numbers along the diagonal of the Smith Normal Form for A as considered above) for $n_1 \geq 2$, then the rank $= r$.*

For example, $\langle \mathbb{Z}, + \rangle \geq \langle 5, 8 \rangle = \mathbb{Z}$, and in general $\langle a, b \rangle = \langle \gcd(a, b) \rangle$, so all subgroups of $\mathbb{Z}$ have rank $= 1$.

Exercise 1.18. *(Recall direct products from June 27)* $C_{35} \cong C_5 \times C_7 = \langle a, b \mid 5a = 0, 7b = 0 \rangle$.

Dhar wrote a significant article about the sandpile group for the $n \times n$-grid, where you add one sink and attach one edge from each points on the perimeter of the grid to the sink, except for the corners which are attached by two edges to the sink (i.e. so that the valence of every vertex remains 4).

For this sandpile, we have $V_0 = \{(i, j) \mid 1 \leq i, j \leq n\}$ and

$$M = \langle t_v : v \in V_0, t_{ij} = t_{i-1,j} + t_{i+1,j} + t_{i,j-1} + t_{i,j+1} \rangle$$

Theorem 1.19. *(Dhar) The rank of this sandpile group is n.*

Exercise 1.20. *Prove that rank$(G) \leq n$ by proving that $t_{i,1}, i = 1, \cdots, n$ generate G.*

Exercise 1.21. *Prove that the mod-2 rank of G for the $n \times n$ grid is $\geq n$. (HINT: The mod-2 rank of G is the rank of $G/2G$. If $G = \langle t_1, \cdots, t_n | A \rangle$ then $G/2G$ is the quotient of $\mathbb{Z}_2^n$ by the row-space of A over $\mathbb{F}_2$. So the mod-2 rank of G is $n - rk_2(A)$.)*

Exercise 1.22. *If A is an integer matrix then $rk_p(A) \leq rk(A)$. Here $rk_p(A)$ is the rank of A mod p (over $\mathbb{F}_p$).*