REU 2005 · Potpourri · Lecture 2

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10 Potpourri 2

10.1 2-Distance set

Recall the definition of *2-distance sets* in $\mathbb{R}^n$:

Definition 10.1. A *2-distance set* in $\mathbb{R}^n$ is a set of points $\{p_1, \ldots, p_m\}$ such that $dist(p_i, p_j) \in \{\alpha, \beta\}$ for $i \neq j$. That is, the distance between any two distinct points in the set has one of two fixed values.

We denote by $m_2(n)$ the maximum size of a *2-distance set* in $\mathbb{R}^n$.

Exercise 10.2. *Prove that* $m_2(n) > cn^2$ *for some constant* c.

Recall that a *one-distance set* in $\mathbb{R}^n$ has $\leq (n+1)$ vertices. Recall that the set of standand unit vectors $\{e_i : 1 \leq i \leq n\}$ is a set of size n and is a *one-distance set in* $\mathbb{R}^n$.

Taking clue from the above example we can say that the set $\{(e_i + e_j) : 1 \leq i < j \leq n\}$ is a *2-distance set* in $\mathbb{R}^n$. The size of this set is $\binom{n}{2}$.

Exercise 10.3. $m_2(n) \geq \binom{n+1}{2}$.

10.2 Linear Algebra

Consult the *Linear Algebra and appications to graphs* Part I and II handouts.

10.3 Adjacency matrix of graph

Let G be a simple graph (that is, no multiple edges and no self loop). Let the adjacency matrix is A. And let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix of G.

In other words the charecteristic polynomial $f_A(x)$ factors as $f_A(x) = \Pi(x - \lambda_i)$.

Exercise 10.4. *Prove that* $\deg_{\min} \leq \lambda_1 \leq \deg_{\max}$, *where* $\deg_{\min}$ *and* $\deg_{\max}$ *are the minimum and maximum degrees of the graph, respectively.*

Exercise 10.5. $\lambda_1 \geq$ average degree. *Here "average degree" is* $\frac{\Sigma_{i\in V}\deg(i)}{n} = \frac{2m}{n}$*, where* m *is number of edges.*

Exercise 10.6. $\Sigma_{i=1}^{n}\lambda_i = 0$.

By the *trace* of a matrix we mean the sum of the diagonal entries.

Exercise 10.7. *For any square matrix the sum of the eigenvalues is same as the trace.*

From this it follows that unless G is the empty graph (no edges) $\lambda_1 > 0$ and $\lambda_n < 0$.

Exercise 10.8. *Prove that* $|\lambda_n| \leq \lambda_1$.

Exercise 10.9. *If* G *is bipartite then prove that* $\lambda_{n-i} = -\lambda_{i+1}$ *for all* $0 \leq i < n$.

Conversely,

Exercise 10.10. *Prove that if* G *is connected and* $\lambda_n = -\lambda_1$ *then* G *is bipartite. Show that this statement fails if we drop the assumption of connectedness.*

Exercise 10.11. *If* M *is a square matrix over* $\mathbb{C}$ *with eigenvalues* $\mu_1, \mu_2, \ldots, \mu_n$*, then the eigenvalues of* M^2 *are* $\mu_1^2, \mu_2^2, \ldots, \mu_n^2$.

The proof of the above exercise is easier when M is real symmetric.

Corollary 10.12. $\Sigma_{i=1}^{n}\mu_i^2 =$ *trace of* M^2.

Exercise 10.13. *Give a combinatorial meaning to* $\Sigma_{i=1}^{n}\lambda_i^2$.
[HINT (1) : The answer is very simple]
[HINT (2) : The above corollary]

Exercise 10.14. *Find the combinatorial meaning of each entry of* A^2.

Definition 10.15. A *regular graph* of degree r is a graph for which the degree of all the vertices is r.

From a previous exercise we get that for a regular graph $\lambda_i = r$.

Exercise 10.16. *Find a eigenvector corresponding to eigenvalue* r *of the regular graph of degree* r.

Exercise 10.17. *What is the multiplicity of the eigenvalue* r*? [HINT: Answer is a very simple combinatorial parameter of* G*.]*

10.4 Interlacing

Exercise* 10.18. *Let G be a general graph with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $H = G \backslash v$ (that is, the graph G with the vertex v deleted) and let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ be the eigenvalues of H. Then,*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$$

This property is called *interlacing.*
We can actually prove a more general statement.

Exercise* 10.19. *Let A be a $n \times n$ real symmetric matrix. Let B is the matrix obtained by deleting the i-th row and i-th column of A. Prove that the eigenvalues of B interlace the eigenvalues of A.*
[HINT (1): Use Spectral Theorem]
[HINT (2): Prove $\lambda_1 \geq \mu_1$ using the following exercise]

Exercise 10.20 (Courant-Fisher Equality). $\lambda_1 = \max_{\|x\|=1} x^T A x$.

10.5 Chebyshev's Polynomial

Consider the following functions:

$$T_n(\cos\theta) = \cos(n\theta)$$

.

Exercise 10.21. *Prove that T_n is a polynomial of degree n. Find its leading coefficient.*

These polynomials are called *Chebyshev's Polynomials of the first kind.*
For example $T_0(x) = 0$.
$T_1(x) = x$.
$T_2(x) = 2x^2 - 1$. This follows from the fact $\cos(2\theta) = 2\cos^2(\theta) - 1$.

Exercise 10.22.

1. *Calculate $T_3(x) and T_4(x)$.*
2. *Give a recursive formula for T_n.*
3. *Show all roots of T_n are real and are interlaced by the roots of T_{n-1}.*

Exercise 10.23. *Calculate the eigenvalues of the*

1. *Path of length $(n-1)$.*
2. *Cycle of length n.*

Exercise 10.24. *Find the connection between the previous two exercises.*