REU 2005 · Potpourri · Lecture 4

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4 Eigenvalues of graphs

4.1 Powers of Adjacency matrix

Definition 4.1 (Adjacency Matrix). Let $G = (V, E)$ be a graph; assume $V = \{1, 2, \ldots, n\}$. The adjacency matrix $A_G = (a_{ij})$of G is the $n \times n$ $(0, 1)$ matrix defined by:

$$a_{ij} = \begin{cases} 1 & i \sim j \\ 0 & i \not\sim j \end{cases}$$

The (i, j) entry of $(A_G)^k$ gives the number of walks of length k between vertex i and vertex j. The (i, i)th entry of $(A_G)^k$ corresponds to the number of closed walks of length k containing vertex i. Hence we get that

$$\mathrm{tr}(A_G^2) = \sum_i \deg_i = 2m = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2$$

where m is the number of edges of G. Similarly we get,

$$\mathrm{tr}(A_G^3) = 6t = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2$$

where t is the number of triangles in G.

4.2 Proof of $\deg_{\min} \leq \lambda_1 \leq \deg_{\max}$

For this section we use the following notation: G is a graph on $\{1, 2, \ldots, n\}$ and A is the adjacency matrix of G. The eigenvalues of A are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

Claim 4.2. $(\forall i)(\lambda_i \leq \deg_{\max})$.

Proof. Let $\underline{0} \neq \underline{x} \in \mathbb{R}^n$ be such that $A\underline{x} = \lambda_i \underline{x}$. Let j be an index such that $|x_j|$ is the maximum.

$$\begin{array}{rcccl} \lambda_i |x_j| & = & |(A\underline{x})_j| & = & |\sum_k a_{jk} x_k| \\ & = & \left|\sum_{k:k\sim j} x_k\right| & \leq & \sum_{k:k\sim j} |x_k| \\ & \leq & \sum_{k:k\sim j} |x_j| & \leq & \deg_{\max} |x_j| \end{array}$$

□

The above proof actually proves a stronger statement. If G is a directed graph and A is the adjacency matrix of G, then the largest eigenvalue λ_1 of A is at most the maximum outgedree of G.

Definition 4.3 (Quadratic Form). The quadratic form associated with a real symmetric matrix A is a function $Q_A(\underline{x}) : \mathbb{R}^n \to \mathbb{R}$, defined by

$$Q_A(\underline{x}) = \underline{x}^T A \underline{x} = \sum_{i,j} a_{ij} x_i x_j$$

Definition 4.4. A set of vectors $a_1, \ldots, a_n$ is orthonormal if $a_i \cdot a_j = \delta_{ij}$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

Theorem 4.5 (Spectral Theorem). *If A is a real symmetric matrix, then all the eigenvalues of A are real and A has an orthonormal eigenbasis.*

If A is a real symmetric matrix, then A has an orthonormal eigenbasis $\underline{e}_1, \ldots, \underline{e}_n$. Then a vector $\underline{x} \in \mathbb{R}^n$ can be written as $\underline{x} = \sum_{i=1}^n r_i \underline{e}_i$. We can now write the quadratic form in a very compact manner.

$$\begin{array}{rcl} Q_A(\underline{x}) & = & \underline{x} \cdot A\underline{x} \\ & = & \left(\sum_{i=1}^n r_i \underline{e}_i\right) \cdot A \left(\sum_{i=1}^n r_i \underline{e}_i\right) \\ & = & \left(\sum_{i=1}^n r_i \underline{e}_i\right) \cdot \left(\sum_{i=1}^n r_i \lambda_i \underline{e}_i\right) \end{array}$$

From the orthonormality of the $\underline{e}_i$s, we get that

$$Q_A(\underline{x}) = \sum_{i=1}^n \lambda_i r_i^2 \tag{4.1}$$

Definition 4.6 (Rayleigh Quotient). If A is a real symmetric matrix, the Rayleigh quotient is the function $R_A(x) : \mathbb{R}^n \to R$ defined by

$$R_A(\underline{x}) = \frac{\underline{x}^T A \underline{x}}{\underline{x}^T \underline{x}}$$

Theorem 4.7 (Courant-Fischer Theorem, baby case). *If A is a real symmetric matrix with the largest eigenvalue λ_1 then,*

$$\lambda_1 = \max_{x \in \mathbb{R}^n} R_A(\underline{x})$$

Proof. We prove this in two steps:

1. $(\exists \underline{x} \in \mathbb{R}^n)(R_A(\underline{x}) = \lambda_1)$.
2. $(\forall \underline{x} \in \mathbb{R}^n)(R_A(\underline{x}) \leq \lambda_1)$.

Item 1 above is easy to see. Let $\underline{x}$ be an eigenvector for λ_1. The Rayleigh quotient for $\underline{x}$ is $R_A(\underline{x}) = \lambda_1$.

Now we prove item 2. Let $\underline{x} = \sum_i r_i \underline{e}_i$ be a vector in $\mathbb{R}^n$ where e_is are the orthonormal eigenbasis for A guaranteed by the spectral theorem. Then by equation 4.1, we know that $Q_A(\underline{x}) = \sum_i \lambda_i r_i^2$. Since $\lambda_1 \geq \lambda_i$, we get that

$$\begin{aligned} Q_A(\underline{x}) &= \sum_i \lambda_i r_i^2 \\ &\leq \sum_i \lambda_1 r_i^2 \\ &= \lambda_1 \sum_i r_i^2 \end{aligned}$$

Since $\underline{x}^T \underline{x} = \sum_i r_i^2$, the statement of the theorem follows. $\square$

Claim 4.8.

$$\lambda_1 \geq \frac{2m}{n}$$

where m is the number of edges in the graph G.

Proof. Due to Courant-Fischer theorem, for all vectors $\underline{x} \in \mathbb{R}^n$, $\lambda_1 \geq R_A(\underline{x})$. We choose a particular vector $\underline{x}$ such that $R_A(\underline{x}) \geq \frac{2m}{n}$.

Let $\underline{\mathbf{1}}$ be the all 1's vector. It is easy to see that $\underline{\mathbf{1}}^T A \underline{\mathbf{1}} = 2m$ where m is the number of edges in the graph. Since $\underline{\mathbf{1}}^T \underline{\mathbf{1}} = n$, we get that $R_A(\underline{\mathbf{1}}) = \frac{2m}{n}$. $\square$

Since $\frac{2m}{n} \geq \text{deg}_{\text{min}}$, it follows that $\lambda_1 \geq \text{deg}_{\text{min}}$.

Exercise 4.9. *Prove that $\lambda_1 \geq \sqrt{\frac{\sum_i (\text{deg}_i)^2}{n}}$.*

Study Question 4.10. *When is $\lambda_1 = \frac{2m}{n}$?*

Study Question 4.11. *When is $\lambda_1 = \text{deg}_{\text{max}}$?*

4.3 Interleaving theorem

Theorem 4.12 (Interleaving Theorem). *If A is an $n \times n$ real symmetric matrix, denote by $A^{(-i)}$, the $(n-1) \times (n-1)$ symmetric matrix obtained by removing the i^{th} row and the i^{th} column of A. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of A and $\mu_1 \geq \cdots \geq \mu_{n-1}$ be the eigenvalues of $A^{(-i)}$. Then,*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \mu_{n-1} \geq \lambda_n.$$

Proof of $\lambda_1 \geq \mu_1$: Without loss of generality assume that we remove that last row and the last column. Let $B = A^{(-n)}$. Let $\underline{y} \in \mathbb{R}^{n-1}$ be such that $B\underline{y} = \mu_1 \underline{y}$. Hence

$$\mu_1 = \underline{y}^T B \underline{y}.$$

Now consider a vector $\underline{x} \in \mathbb{R}^n$ obtained by adding 0 as the nth coordinate to $\underline{y}$. That is

$$\underline{x} = \begin{pmatrix} \underline{y} \\ 0 \end{pmatrix}.$$

Notice that $\underline{x}^T A \underline{x} = \underline{y}^T B \underline{y}$ and $\underline{x}^T \underline{x} = \underline{y}^T \underline{y}$. Hence by Courant-Fischer Theorem,

$$\lambda_1 \geq R_A(\underline{x}) = R_B(\underline{y}) = \mu_1$$

□

To prove the other inequalities, we need the full power of Courant-Fischer theorem.

Theorem 4.13 (Courant-Fischer Theorem, general case). *If A is a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ then,*

$$\lambda_i = \max_{\substack{\mathcal{U} \leq \mathbb{R}^n \\ \dim(\mathcal{U}) = k}} \min_{\underline{x} \in \mathcal{U}} R_A(\underline{x})$$

Exercise 4.14. *Prove the Courant-Fischer theorem (Theorem 4.13).*
*(*Hint: *Use the spectral theorem)*

Exercise 4.15. *Use the Fischer-Courant Theorem (Theorem 4.13) to prove the interleaving theorem.*

4.4 Isoperimetric Inequality

Definition 4.16 (Laplacian). We define the Laplacian of the graph to be

$$L_G = D_G - A_G$$

where A_G is the adjacency matrix and D_G is a diagonal matrix $D_G(i,i) = \deg_G(i)$. Let the eigenvalues of L_G be $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$.

Exercise 4.17. *Prove that* $\lambda_0 = 0$.

Exercise 4.18. *Prove that* $\lambda_1 > 0 \iff G$ *is connected.*

Definition 4.19 (Iso-perimetric ratio). If $A \subseteq G$ we denote by $\delta(A)$ the number of edges between A and $A = V \setminus A$. The isoperimetric ratio for A is $\frac{\delta(A)}{|A|}$.

The next result shows the important fact that if λ_1 is large then G is highly expanding".

Exercise 4.20. *Prove that* $\frac{|\delta A|}{|A|} \geq \frac{\lambda_1}{2}$