2005 REU Problems: The Rest

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1. A random graph on n vertices can be obtained in the following way. The vertex set of the graph is $\{1, \ldots, n\}$ and we draw all the possible (undirected) edges independently with probability 1/2 (in other words, for all the $\binom{n}{2}$ possible edges toss a coin and draw the edge if you got heads). The same way you can define a random graph on countably many vertices. Show that there is only ONE countable random graph: that is, almost all random graphs on a countable set are isomorphic.

2. Is there a measurable subset $A \subseteq [0, 1]$ such that for every nonempty open interval $I \subseteq [0, 1]$ both $I \cap A$ and $I \backslash A$ have positive measure?

3. Does there exist continuously many subsets of $[0, 1]$, all of measure 1/2, such that any two different sets have intersection of measure 1/4?

4. Can you cover the set of nonnegative integers with finitely many disjoint arithmetic progressions of distinct differences?

5. Let $F = F_2$ be the free group on two generators. Show that for every $g \in F$ the intersection of finite index subgroups of F containing g equals the cyclic group generated by g.

6. For subsets $A, B \subseteq \mathbb{Z}$ let $A + B = \{a + b \mid a \in A, b \in B\}$. For $A \subseteq \mathbb{N}$ let us define the upper density as

   $$\overline{d}(A) = \lim_{n\to\infty} \sup \frac{|A \cap [1, n]|}{n}$$

   Is there a subset $A \subseteq \mathbb{N}$ of zero upper density such that $A + A = \mathbb{N}$?

7. Are there 4 points on the plane such that all 6 possible distances are odd integers?

8. Show that $PSL(2, \mathbb{Z}) \cong C_2 * C_3$.

9. Prove that every large enough alternating group can be generated by an element of order 2 and an element of order 3.

10. Show that a 3-regular graph on n vertices has at most $2^{n/2+1}$ cycles.

11. Prove that the real function $f(x) = x^2$ can be obtained as the sum of three periodic real functions.

12. Assume that a_n and b_n are positive non-decreasing sequences such that both $\sum_n 1/a_n$ and $\sum_n 1/b_n$ are divergent. Does this imply that

$$\sum_n \frac{1}{a_n + b_n}$$

is divergent?

13. Are there two proper subsets A, B of the plane such that for any $A' \backsim A$ and $B' \backsim B$ we have $|A' \cap B'| = 2005$?

14. Show that if A is any set of points of size $d+2$ in R^d then there exists a partition $A = B \cup C$ such that the convex hull of B intersects the convex hull of C.

15. Let G be a finitely generated group. Show that for every integer n there are finitely many subgroups of G of index n.

16. Is there a continuous function on $[0, 1]$ which takes on every value in its range countably infinitely many times?