REU 2006 · Apprentice · Lecture 3b

Instructor: László Babai
Scribe: Deepam Patel
Editor: Sourav Chakraborty

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These are the lecture notes for the 2nd half of the Apprentice class on June 28, 2006.

3b.1 Binomial Theorem

Definition 3b.1.1. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$

Throughout p will be a prime.

Exercise 3b.1.2. *If p is prime, $1 \le k \le p-1$ then $p \mid \binom{p}{k}$.*

Theorem 3b.1.3 (Binomial Theorem).

$$(x+y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k y^{n-k}. \tag{3b.1.1}$$

If you haven't seen a proof of the Binomial Theorem then prove it as an exercise.

Exercise 3b.1.4. *Let a and b be integers. Then $(a+b)^p \equiv a^p + b^p \pmod{p}$.*

Exercise 3b.1.5. *Use the preceding exercise to prove Fermat's little Theorem.*

As a consequence of the binomial theorem one has the following identity:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = (1+1)^n = 2^n. \tag{3b.1.2}$$

We can also give a combinatorial proof of this identity. Let A be a set of size n. Consider the subsets of size k of A, here $0 \le k \le n$. This sum represents the left side of the above identity. On the other hand, giving a subset of A is equivalent to assigning 0 or 1 to each $1 \le k \le n$. The 0 or 1 tells us whether or not the given element is in the subset. There are 2^n such choices, giving the right half of the identity.

Let us say that a set is *even* if it has an even number of elements; and *odd* if it has an odd number of elements.

Let us now count the even subsets of A. One might guess that this number is half the total number of subsets. Indeed, it is 2^{n-1}. We shall give two proofs of this fact.

Observe that the number of even subsets is $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}$.

Theorem 3b.1.6. *For all $n > 0$, the number of even subsets of A is equal to the number of odd subsets of A.*

Proof. Note that the number of odd subsets is $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k+1}$. Applying the binomial theorem we get that that the alternating sum of the binomial coefficients is $0 : \binom{n}{0} - \binom{n}{1} + \cdots \pm \binom{n}{n} = (1-1)^n = 0^n = 0$. Note, here we use the fact that $n > 0$ as by our convention $0^0 = 1$. □

When n is odd we can give an explicit bijection between the even subsets and the odd subsets. If S is an even subset then its complement is odd and vice versa. So for odd n, complementation provides a bijection between even and odd subsets. The next exercise asks find a bijection that works for all $n > 0$.

Exercise 3b.1.7. *If A is a nonempty set then give a bijection between the even subsets and the odd subsets of A.*

Exercise* 3b.1.8. *Let $N(n,3)$ denote number of subsets of A whose size is divisible 3. Then $\left|N(n,3) - \frac{2^n}{3}\right| < 1$.*

3b.2 Chebyshev's Theorem

Now we recall the Prime Number Theorem. Recall $\pi(x)$ denotes the number of primes less than or equal to x.

Theorem 3b.2.1 (Prime Number Theorem). $\pi(x) \sim \frac{x}{\ln(x)}$.

A weaker version of this theorem was proved by Chebyshev:

Theorem 3b.2.2 (Chebyshev's Theorem). *There exist positive constants c_1 and c_2 such that $c_1 \frac{x}{\ln(x)} < \pi(x) < c_2 \frac{x}{\ln(x)}$ for all $x \geq 2$.*

While no simple proof of the Prime Number Theorem is known, we shall prove Chebyshev's theorem in the next class. We list a series of exercises geared towards this proof.

Exercise 3b.2.3. $\frac{4^n}{2n+1} < \binom{2n}{n} < 4^n$.

Exercise 3b.2.4. $\binom{2n+1}{n} < 4^n$.

Exercise 3b.2.5. *Find the exponent of the prime p in $n!$ (i. e., find the largest k such that $p^k \mid n!$).*

Exercise 3b.2.6. *Prove: if p^ℓ divides $\binom{n}{k}$ then $p^\ell \leq n$.*

Exercise* 3b.2.7. *Show that* $\prod_{p \le x} p \le 4^x$.

Hint: Observe that

$$\prod_{k+2 \le p \le 2k+1} p \;\Big|\; \binom{2k+1}{k}. \tag{3b.2.1}$$

Exercise 3b.2.8. *Use Exercise 3b.2.7 to prove the upper bound portion of Chebyshev's theorem: there exists* $C > 0$ *such that* $\pi(x) < C\frac{x}{\ln(x)}$.

Finally, a couple of unrelated exercises.

Exercise 3b.2.9. $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

Exercise 3b.2.10. *(Experimental exercise.) Draw a large chunk of the Pascal triangle mod 2. Observe the pattern and make conjectures. Prove some of your conjectures.*