

## ABSTRACTS: 2006 SUMMER VIGRE REU

All concepts mentioned in the abstracts will be carefully defined

### APPRENTICE PROGRAM

Apprentices attend courses offered in the full program during week 1. They are strongly recommended to attend the first week of both Babai's and Abert's courses. Those first week materials have been especially designed to be accessible to all participants and include prerequisites for the later apprentice courses.

#### NUMBER THEORY

##### Weeks 2–4: Laci Babai

We shall cover fun topics not usually discussed in abstract algebra, including diophantine approximation and the geometry of numbers, arithmetic functions, partitions of integers, problems in combinatorial number theory. Asymptotic behavior (rates of growth) will often be the subject of study.

#### LINEAR ALGEBRA

##### Weeks 2–4: Miklos Abert

Linear algebra is at the foundation of virtually all areas of mathematics and science. This course will introduce the basics of linear algebra, including the Spectral Theorem, with amusing applications ranging from combinatorics and coding theory to mechanics and statistics. Emphasis will be put on the geometric nature of the subject. The treatment will differ considerably from standard courses.

### FULL PROGRAM

1. Laci Babai. Discrete mathematics. Weeks 1–8.
- 2a. Miklos Abert. Infinite groups and problems. Weeks 1–4.
- 2b. Sinan Unver. Rational points on elliptic curves. Weeks 5–6.
- 2c. Matthew Kerr. Introduction to complex algebraic curves. Weeks 7–8.
- 3a. Robert Fefferman. Some ideas in pure and applied analysis. Weeks 1–2.
- 3b. Thomas Fiore. Mathematics in music theory. Weeks 3–4.
- 3c. Laura DeMarco. Fractals and dimension. Weeks 5–6.
- 4a. Benson Farb. Some basic open problems in topology. Weeks 1–4.
- 4b. Juan Souto. Geometry of Riemann surfaces. Weeks 5–6.
- 4c. Uri Bader. Random walks on graphs. Weeks 7–8.
5. Peter May.  $K$ -theory. Weeks 1–2, 5–8.

## 1. DISCRETE MATHEMATICS

### Weeks 1–8: Laci Babai

The course covers topics in number theory, combinatorial structures, linear algebra, discrete probability, finite groups, the theory of algorithms, and combinatorial models in the theory of computing. The course will highlight surprising interactions between these areas. Students will discover each field through solving sequences of challenging problems. A number of open problems will also be discussed.

The course will be divided into two modules. In the first module (weeks 1–4) we shall discuss selected topics in number theory, combinatorics, linear algebra, discrete probability, finite groups, the Discrete Fourier Transform, with a focus on the interactions between these fields.

The first week will cover topics in number theory and will complement Miklos Abert's first week, also dedicated to number theory.

The second module (weeks 5–8) will highlight the beautiful mathematics of the theory of computing, including devilishly clever algorithms as well as applications of number theory and linear algebra to prove lower bounds in combinatorial models of computation.

The two modules will be sufficiently independent so if you missed the first module, you can still join the second.

Returning students will not be bored.

PQ: Consent of instructor. CMSC-27100 (Discrete Math) or CMSC-27000 (Algorithms) helpful but not required. Basic linear algebra and finite fields desirable.

## 2a. INFINITE GROUPS AND PROBLEMS

### Weeks 1–4: Miklos Abert

The first week will be introduction to number theory, accessible on all levels. It provides vital background for the YSP, some background to Babai's second–fourth weeks and my second–fourth weeks.

The second, third and fourth weeks will be about (mostly infinite) group theory, with special emphasis on groups acting on trees. I will present some very strange groups naturally acting on rooted trees and also build a beautiful theory of groups acting on unrooted trees. Some knowledge of algebra may help, but I will define everything and it will be easy to catch up. The course will also have a problem solving aspect; we will present and discuss problems from exercise to research levels.

2b. RATIONAL POINTS ON ELLIPTIC CURVESWeeks 5 and 6: Sinan Unver

An elliptic curve is the set of solutions of a cubic polynomial in two variables. When the polynomial has rational coefficients one can ask for a description of the rational solutions. This number theoretic question will be our main interest in this course.

In order to arrive at this description we will start with the geometry of elliptic curves. Then we will study elliptic curves over real and complex numbers and over finite fields. We will end with the concepts of height and descent and Mordell's theorem. These topics will include beautiful ideas from geometry, analysis and algebra. Finally, if time permits, we will end with a conjecture on elliptic curves that is equivalent to a deep conjecture on rational points on higher genus curves. In this sense a satisfactory understanding of rational points on elliptic curves is connected to the main problems in diophantine geometry.

2c. INTRODUCTION TO COMPLEX ALGEBRAIC CURVESWeeks 7 and 8: Matthew Kerr

The ostensible topic here is the solution sets of equations in two variables in the (complex projective) plane, a description which belies both the considerable beauty of the subject and the considerable machinery which is usually part of its study. I intend to avoid the latter by focusing mainly on equations of degree 2 (conics) or 3 (elliptic curves) and sticking to a very classical, geometric treatment with plenty of pictures. Linear algebra and several-variable calculus are prerequisites; having some acquaintance with differential forms, algebra and complex analysis would be very helpful.

I will start with the (ancient) problem of how to construct the conic through five given points in the plane, and Pascal's elegant solution involving inscribed hexagons. Elliptic curves arise naturally from a problem of a similar flavor, involving polygons inscribed in one conic and circumscribing another. We will also see how both kinds of curves arise from certain (trigonometric and Abelian) multivalued integrals.

A brief introduction to the projective plane will then lead us to the group structure on the points of an elliptic curve, and its relation (called Abel's theorem) to the above integrals and to meromorphic functions on the curve. My goal is for the class to at least understand how the statement of Abel generalizes to other algebraic curves, and to touch on a few other important topics – for example uniformization, theta functions, and Picard-Fuchs differential equations – along the way.

### 3a. SOME IDEAS IN PURE AND APPLIED ANALYSIS

#### Weeks 1 and 2: Robert Fefferman

This will be a discussion of how a number of different ideas in pure and applied analysis come together in a very surprising way. The ideas I have in mind are:

1. Convergence of Fourier Series
2. Gambling Games
3. The Fundamental Theorem of Calculus
4. Maximal Functions
5. Wavelets

### 3b. MATHEMATICS IN MUSIC THEORY

#### Weeks 3 and 4: Thomas Fiore

Have you ever wondered if the relationship between mathematics and music extends to the composition and interpretation of musical works? If you have, then this course is for you.

A central concern of music theory is to find a good way of hearing a piece of music and to communicate that way of hearing. Music theorists often draw upon mathematics to create conceptual categories towards this end. In recent years, basic tools from group theory, combinatorics, and topology have entered the realm of musical analysis, especially in the work of David Lewin and Rick Cohn.

We will explore these developments by drawing on examples from Bach, Beethoven, Wagner, Hindemith, and the Beatles. For mathematicians, one of the most surprising discoveries was perhaps the presence of a certain topological structure in Beethoven's Ninth Symphony. This is especially surprising when one considers that Beethoven composed this work over 70 years before Henri Poincare initiated the study of topology! This topological structure will be revealed in the course of the lectures.

The ability to read music is not a prerequisite, since we aim to see and hear mathematics in action.

### 3c. FRACTALS AND DIMENSION

#### Weeks 5 and 6: Laura DeMarco

You've heard of fractals, but what are they? What is an object of dimension  $3/2$  or of dimension equal to the square root of 2? We'll study self-similar sets and notions of fractional dimension and how they arise naturally in analysis and dynamics.

4a. SOME BASIC OPEN PROBLEMS IN TOPOLOGYWeeks 1–4: Benson Farb

The purpose of this four week module will be to throw students into exploring some fundamental but open problems in topology. Some decent percentage of the time will be spent in groups discussing ideas for solutions of problems. One might think of these as "research teams"; they will be directed by graduate students.

I will probably concentrate on problems about triangulations. One example: fix a manifold  $M$ , for example a torus or surface of genus  $g$  or an  $n$ -dimensional sphere. Let  $f(k)$  be the number of (combinatorial types of) triangulations of  $M$  with  $k$  simplices.

Question: What are the asymptotics of  $f(k)$  as  $k \rightarrow \infty$ ?

For example, for the  $n$ -sphere one can easily prove that  $f(k)$  grows more slowly than  $e^{k^2}$ . The guess is that  $e^k$  is the right order of growth, but this is still open. While I am interested in this problem because it seems fundamental and really interesting, others might find motivation from the fact that the  $e^k$  growth is apparently wanted to make various models of quantum gravity work.

Prerequisites: basically nothing. Knowing basic topology, what a manifold is, and some basic group theory would be useful, though.

4b. GEOMETRY OF RIEMANN SURFACESWeeks 5 and 6: Juan Souto

In this two week mini-course we will study various topics in the theory of Riemann surfaces. Riemann surfaces are at the meeting point of algebraic geometry, complex analysis, dynamics, differential geometry, topology...

After discussing some basic facts about the geometry of Riemann surfaces of genus  $g \geq 2$  we will show that the automorphism group of such a surface has at most  $84(g-1)$  elements. Once this is done we will study the dynamics of the geodesic and horocyclic flows. If time allows, we will also describe the Teichmüller space of the torus and its relation to hyperbolic geometry.

Most arguments are of a geometric nature and are accessible to second (or even first) year mathematics students.

4c. RANDOM WALKS ON GRAPHSWeeks 7 and 8: Uri Bader

In this two weeks mini-course we will study various topics in the theory of random walks on graphs. By considering the simple setting of a graph endowed with transition probabilities attached to each edge, we will have the opportunity to meet and study some of the basic ideas involved in Dynamical Systems, the main tool being a rather naive Linear-Algebra.

The theory of random walks, while being accessible to second (or even first) year mathematics students, is at the meeting point of Probability, Dynamical Systems, Combinatorics, Information Theory and Geometry. It is a well established theory and has a variety of real world applications ("Google" is just one example), yet it is still a lively field of research in modern mathematics.

## 5. $K$ -THEORY

Weeks 1–2 and 5–8: Peter May

Loosely speaking, I mean by  $K$ -theory anything that starts with the idea of constructing algebraic structures by grouping together “Klasses” (the German spelling) of objects of a given kind. The starting constructions are actually very naive, a generalization of how one constructs the integers from the natural numbers or the rational numbers from the integers.

The first two weeks will give some basic examples, such as the Burnside ring  $A(G)$ . This is a ring that is constructed from isomorphism “Klasses” of finite sets with an action of a given finite group  $G$ , and sample calculations are fun.

The last four weeks will break into two segments, the second largely repeating the first. Weeks 5 and 6 are scheduled while those in YSP are working and weeks 7 and 8 are scheduled while those in SESAME are working. *This is the only class in the REU that is scheduled during either YSP or SESAME working hours.*

The aim of these two week mini-courses is to explain in outline how topological  $K$ -theory works to prove that the only real division algebras are the real numbers, the complex numbers, the quaternions, and the octonions. In odd dimensions greater than 1 and in even dimensions greater than 8, there are no real division algebras. The essential point is the beautiful and surprising interaction between various branches of mathematics.