

REU 2006 · Discrete Math · Lecture 6

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1 Symmetries of Regular Polyhedra

Last time, we talked about groups of symmetries of regular polyhedrons, and made some statements which should be tantalizing.

For instance, we said that

$$\text{Aut}(\text{Petersen's graph}) \cong S_5,$$

and that

$$\text{Aut}^+(\text{Cube}) \cong S_4,$$

where the $+$ means “orientation-preserving.” Most tantalizing of all, we said

$$\text{Aut}^+(\text{dodecahedron}) \cong A_5.$$

Remark: one can use this last fact as a step in the proof that the roots of general quintic polynomials cannot be expressed with radicals.

1.1 The Cube

So, how do we prove these statements? Start with orientation-preserving symmetries of the cube. A symmetry permutes the vertices (8 objects) and

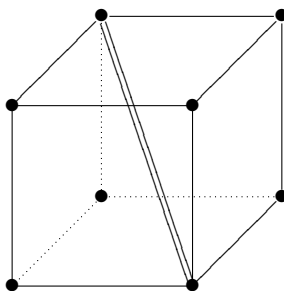


Figure 1: A cube, with one main diagonal marked as a double line

the faces (6 objects). The center of the cube is always preserved. Since we are talking about orientation-preserving symmetries, lines through the center must get sent to lines through the center.

So, consider the four “main diagonals” of the cube, connecting opposite corners (see Figure 1). If we apply any symmetry of the cube, these four diagonals are permuted. Thus we get a map $\text{Aut}^+(\text{cube}) \rightarrow S_4$ (and $\text{Aut}(\text{cube}) \rightarrow S_4$) by labeling the diagonals and sending a symmetry to the appropriate permutation on 4 objects. It is easy to see this map is a homomorphism. What are some of the elements of S_4 which we get in this way? Well, a rotation by $\frac{2\pi}{3}$ about diagonal #1 fixes 1 and gives a 3-cycle on the remaining diagonals, i.e. we get the element $(1)(234)$ of S_4 . We could go through the elements of S_4 and tediously check whether we get other elements.

What symmetries get sent to the identity? Well, first look at which (not necessarily orientation-preserving) symmetries get sent to the identity of S_4 .

Exercise 1. Show that the only symmetries of the cube which induce the identity permutation on the set of the 4 main diagonals are the identity and the map which sends every point x to $-x$ (viewing the center as the origin).

Thus the kernel of the map $\text{Aut}(\text{cube}) \rightarrow S_4$ has size 2, so this is a 2-to-1 map. We haven’t proved the map is onto, but we know from last time that $|\text{Aut}(\text{cube})| = 48$. By the exercise, this means that $\text{Aut}^+(\text{cube}) \rightarrow S_4$ is a bijection between two sets of size 24. So it must be an isomorphism.

Alternately, we could prove directly that the map is an isomorphism by showing that it is onto (the exercise shows that it is one-to-one). To do this,

we just need to pick a set of generators of S_4 and show that they are in the image. For instance, (12) is in the image. Pick the line through the center which passes through the midpoints of the edges connecting diagonal 1 to diagonal 2. With a little thought, you can see that a 180° rotation about this line gives (12) . Since (12) and (234) generate S_4 , this means that the map is onto.

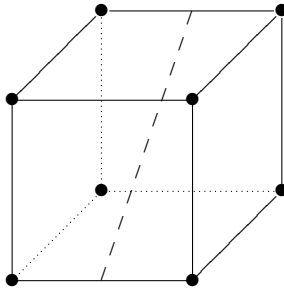


Figure 2: Rotating about the dashed line gives the permutation (12)

1.2 The Dodecahedron

Exercise 2. Show that $\text{Aut}^+(\text{dodecahedron}) \cong A_5$. Hint: pick a set of 5 objects which are permuted by symmetries of the dodecahedron, and show that the map from symmetries of the dodecahedron to permutations on these 5 elements gives all even permutations. The following exercises may help.

Exercise 3. Show that A_n is generated by all 3-cycles (for $n \geq 3$).

Exercise 4. Show that A_n is generated by all permutations of type $2 + 2$ for $n \geq 5$ (that is, permutations like $(12)(34)$).

1.3 Petersen's Graph

Finally, let's move on to Petersen's graph. Here we don't have a geometric representation, so we'll have to try a different approach. Notice that there are 10 vertices of degree 3 each, so there are $15 = 3 \cdot 5$ edges. Maybe we can split up the set of edges into 5 triples, such that any isomorphism of the graph takes a triple into another triple. We can take the triple shown in Figure 3, plus the 4 other triples you get by rotating.

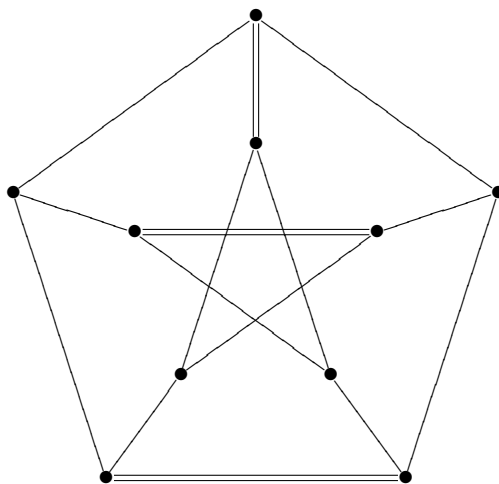


Figure 3: Petersen's graph with a triple of edges marked

There are lots of things you need to prove about this.

Exercise 5. This partition of the 15 edges into 5 triples is invariant under isomorphisms of the graph. In other words, the partition is preserved by all automorphisms.

Once we have proved this exercise, we get a homomorphism $\varphi : \text{Aut}(\text{Petersen's graph}) \rightarrow S_5$, by looking at how an automorphism permutes the triple.

Exercise 6. Show that $|\text{Aut}(\text{Petersen})| = 120$. Hint: use the triples.

Once we have this, then if we can show that φ is onto, then we will be done (an onto map from a group of size 120 to a group of size 120 must be an isomorphism).

Exercise 7. Find an automorphism which switches the two triples marked in Figure 4, and fixes all others.

By symmetry, this shows that all transpositions are in the image, and so φ is an isomorphism.

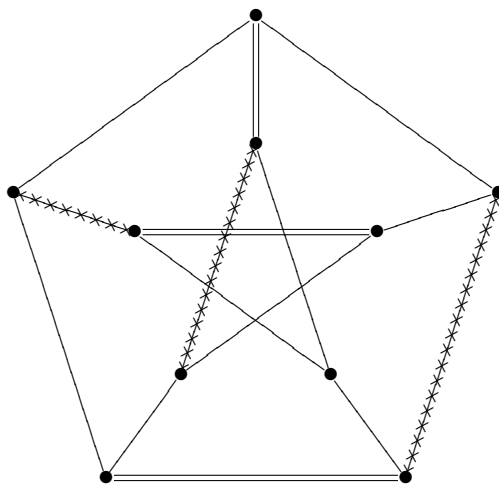


Figure 4: Petersen's graph with two triples of edges marked

This is not an ah-ha! proof. It takes a lot of work. We will now do an ah-ha! proof, which will actually shed some light on this outline.

Pick 5 points, labeled 1, 2, 3, 4, 5. Notice that $\binom{5}{2} = 10$, which suggests trying to label the vertices of Petersen's graph with pairs of numbers between 1 and 5. One way of doing this is shown in Figure 5.

Notice that, with this labeling, vertex $\{a, b\}$ and vertex $\{c, d\}$ are connected if and only if $\{a, b\} \cap \{c, d\} = \emptyset$. This immediately proves what we want—every permutation of $1, 2, \dots, 5$ induces a permutation of pairs which induces an automorphism of Petersen's graph. For example, the permutation (12) fixes the vertex $\{1, 2\}$, switches the three pairs $\{1, x\}$ and $\{2, x\}$ for $x = 3, 4, 5$, and leaves the vertices $\{x, y\}$ with $x, y \in \{3, 4, 5\}$ alone. You can check that this is an automorphism of Petersen's graph. This is interesting—notice that the path from $\{4, 5\}$ to $\{1, 2\}$ to $\{3, 4\}$ to $\{5, 1\}$ gets sent to the path from $\{4, 5\}$ to $\{1, 2\}$ to $\{3, 4\}$ to $\{5, 2\}$. That is, if you have any path of length 3, there is an automorphism which fixes the first 2 edges and moves the last one. It follows that, given any two paths of length 3 in Petersen's graph, there is an automorphism which takes the first path into the second.

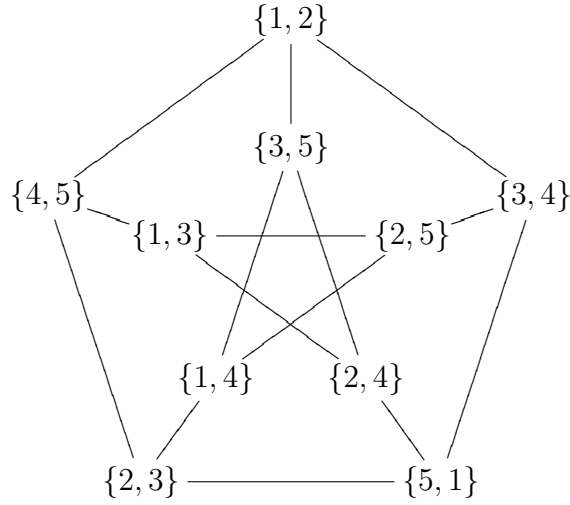


Figure 5: Petersen's graph with vertices labeled

2 Kneser's graphs

Petersen's graph is a special type of a **Kneser graph**.

Definition 8. A **Kneser graph** is a graph with $\binom{n}{k}$ vertices, labeled by the k -subsets of an n -set (the subsets of size k of a set of size n), and adjacency given by connecting disjoint subsets.

Notice that, if $n < 2k$, then there are no disjoint k -sets, so the graph has no edges. If $n = 2k$, then each vertex has degree 1. So in order for a Kneser graph to be interesting, we must have $n \geq 2k + 1$.

3 More graph theory

Definition 9. The **girth** of a graph is the length of the shortest cycle.

For example, Petersen's graph has girth 5.

Example 10. If G has girth 5 and every vertex has degree r , then $n > r^2 + 1$.

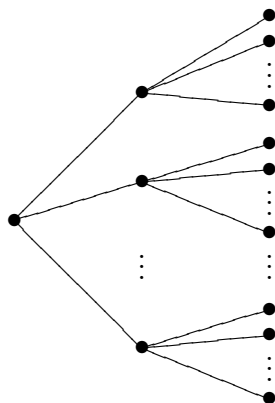


Figure 6: Subgraph of a graph with girth r

Proof. Pick a vertex. It is connected to r vertices. None of these can be connected to each other, because then we would have 3-cycles. Each of these r vertices is connected to $r - 1$ new vertices, and again there can be no other connections among these, because then we would have 3-cycles or 4-cycles (see Figure 6). So the total number of vertices is at least $1 + r + r(r - 1) = r^2 + 1$. \square

The interesting thing is that, in Petersen's graph, we have exactly $n = r^2 + 1$.

Exercise 11. If $n = r^2 + 1$, then G has diameter 2.

Theorem 12. Under these conditions, if $n = r^2 + 1$, then $r \in \{2, 3, 7, 57\}$.

We will prove this later.

Remark. For $r = 7$, $n = 50$, Hoffman and Singleton constructed the remarkable Hoffman-Singleton graph, which gives an example of such a graph for this n, r . They made it by pasting together copies of Petersen's graph. People have been looking for a long time, but it is still unknown whether there exists a graph of this type with $r = 57$.

4 Graph Coloring

Definition 13. A legal coloring of a graph G is a function from the set $V(G)$ of vertices to a set of colors such that adjacent vertices have different colors.

Definition 14. The **chromatic number** $\chi(G)$ of a graph G is the minimum number of colors required for a legal coloring.

Example 15. For example, $\chi(C_5) = 3$. (If you try to color it with 2 colors, then a simple argument shows you wind up with 2 adjacent vertices of the same color.) Also,

- $\chi(K_n) = n$
- $\chi(K_{n,n}) = 2$
- An empty graph (one with no edges) has chromatic number 1.

Definition 16. G is **k -colorable** if $\chi(G) \leq k$.

It is easy to show a graph is k -colorable; you just find a coloring. It is harder to show a graph is not k -colorable.

Exercise 17. Construct a graph G such that G is not 3-colorable, but G contains no triangles (copies of K_3). It should be fairly small. The smallest such has 11 vertices and 5-fold symmetry, and is called Grötsch's graph. Make sure you *prove* that your graph is not 3-colorable.

Exercise 18. If every vertex of a graph G has degree $\leq r$, then $\chi(G) \leq r+1$.

This is a rather easy exercise, but we will upgrade it a little.

Definition 19. A **directed graph**, or **digraph**, is a graph in which every edge is assigned a direction. Given two vertices x and y , there may be two edges connecting x and y (one going x to y , the other going y to x). In a directed graph, there are two notions of degree, the **in degree** (number of edges coming in) and the **out degree** (number of edges going out).

Exercise 20. If G is a digraph, such that $(\forall x \in V(G))(\deg^+(x) \leq r)$, then $\chi(G) \leq 2r + 1$. Here \deg^+ indicates the outdegree. (\deg^- is the indegree.)

Notice that a graph is bipartite if and only if it is 2-colorable.

Exercise 21. A graph G is 2-colorable if and only if G has no odd cycles. (The “only if” direction, G is 2-colorable only if it has no odd cycles, is obvious.)

Exercise 22. Let G be a graph with m edges. Then one can delete $\leq \frac{m}{2}$ edges so that the remaining graph is bipartite. (Hint: use the additivity of expectation.)

Exercise 23. If G is triangle-free, then $\chi(G) \leq 2\sqrt{n} + 1$.

Definition 24. A **tournament** is an oriented complete graph. That is, it is a complete undirected graph in which each edge has been given an orientation.

For example, a tournament might model a sports tournament in which each player has played every other player and there are no ties.

Suppose you want to award one gold medal to the “best” player. This is hard, because there can be cycles (A beats B, B beats C, but C beats A).

Now, what if you want to award two gold medals? Can there be an “embarrassing” tournament, where for every pair of players, there exists a player who has beaten both of them?

- Exercise 25.**
1. Find a 2-embarrassing tournament, i.e. one where every pair is beaten by someone.
 2. Prove that, for every k , there is a k -embarrassing tournament, i.e. one where every k -tuple was beaten by someone.