Problem Sheet 1

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1. There are 17 weights with the property that if you omit any of them you can divide the rest into two equal sized groups, such that the sum weights of the two groups are equal. Show that all the weights are equal.

2. Let S be an infinite set of natural numbers of the form $2^a 3^b$. Show that there exist two distinct elements of S such that one divides the other.

3. A lattice point is a point in the plane with integer coordinates. Let T be a triangle whose vertices are lattice points such that T does not contain any other lattice points, even on its boundary. Show that T has area $1/2$.

4. A necklace is an arrangement of n beads around a circle. We have two kinds of beads, red and blue. Other than their color, the beads are identical. Two necklaces do not count as distinct if one is obtained from the other by rotation.

   Determine the number of necklaces made of n beads where

   (a) $n = p$ is a prime number;

   (b) $n = p^2$ where p is a prime number;

   (c) $n = pq$ where p, q are distinct primes;

   (d) $n = pqr$ where p, q, r are distinct primes.

   Generalize your answers to the case when the beads come in k colors. In each case, your answer should be a simple closed-form expression (no summation symbols or dot-dot-dots).

5. Let n be a positive integer. Show that from any n integers you can choose some such that their sum is divisible by n.

6. A man sells flour at the market. All he has is a balance, a 1 pound weight, a lot of flour and some sacks. However, the balance got inaccurate and now gets balanced exactly when the left and right trays contain weights of some fixed proportion $\lambda \neq 1$ (λ is unknown). Can he still measure 1 pound sacks of flour?

7. The Fibonacci numbers f_i $(i > 0)$ are defined as follows. Let $f_1 = 1$, $f_2 = 1$ and for $n > 2$ let $f_n = f_{n-1} + f_{n-2}$. Show that for every natural number n there exists a Fibonacci number that is divisible by n.

8. Can you cover a 100×100 table with 8×1 "dominoes"?

9. You are given a pair of integers (a, b). A step is to add an integer multiple of one of the entries to the other entry. Can you always reach $(0, *)$ in at most 100 steps?

10. Four frogs stand at the vertices of a square. Without warning they start jumping over each other's heads wildly. More precisely, in every step one of the frogs reflects itself across another frog. Is it possible that after some number of jumps they form a bigger square?

11. Can you cover the plane with disjoint circles of positive radius?

12. Show that every sequence of $n^2 + 1$ distinct real numbers contains an increasing or a decreasing subsequence of length $n + 1$.

13. Assume that a polynomial f maps rationals to rationals. Show that f has rational coefficients.

14. Are there two subsets A and B of the nonnegative integers such that both A and B have at least 100 elements and every nonnegative integer can be uniquely written as the sum of an element of A and an element of B?

15. Are there two infinite subsets A and B of the nonnegative integers such that every nonnegative integer can be uniquely written as the sum of an element of A and an element of B?

16. Prove that the set
$$\left\{n + m\sqrt{2} \mid \ n, m \in \mathbb{Z}\right\}$$
is dense in the real line.

17. Someone painted the plane with 2 colors. Prove that there is necessarily a rectange such that all vertices have the same color.

18. Consider the 8×8 chessboard. Some of the 64 cells are infected. If a cell has at least 2 infected neighbours it becomes infected. (Two cells are neighbors if they share a side.) An infected cell is never cured. Show that you cannot infect the full board with fewer than 8 initially infected cells.

19. Show that for every natural number n the equation
$$\sum_{i=1}^{n} \frac{1}{a_i} = 1$$
has only a finite number of solutions in natural numbers a_i.

20. What is the density of square-free integers? That is, let s_n denote the number of square-free integers between 1 and n. Find the limit
$$\lim_{n \to \infty} \frac{s_n}{n}?$$