REU 2006 Discrete Mathematics Problem sheet

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to be discussed in class on Friday, August 11, 2006

PROBLEMS RELATED TO CLASS MATERIAL

1. (**Erdős–Rényi's unique infinite random graph**) Alice and Bob each construct an infinite graph as follows. The set of vertices is the set of positive integers. Adjacency is decided by independent coin flips. (Infinitely many coins are flipped.) Prove: the probability that the graph constructed by Alice is isomorphic to the graph constructed by Bob is 1.

2. (**Gale's Switching Game**) Player 1 chooses an $n \times n$ matrix M with ± 1 entries. Player 2 takes M and switches the sign of any row or column and repeats this step a finite number of times. ("Switching the sign of a row" means multiplying the entire row by -1.) Let S be the sum of the entries of the resulting matrix. The "payoff" (the amount paid by Player 1 to Player 2) is $|S|$. (Player 1 wants to minimize $|S|$, Player 2 wants to maximize it.) Prove that the value of the game (to Player 2) is $\Theta(n^{3/2})$. (This is how much Player 2 should be willing to pay Player 1 upfront for the privilege of playing the game.) In other words, there exist positive constants c_1 and c_2 such that

   (a) Player 1 can choose a matrix M such that regardless of the moves made by Player 2, $|S| \le c_1 n^{3/2}$;

   (b) regardless of the matrix chosen by Player 1, Player 2 is able to change the signs of appropriate rows and columns so that in the end, $|S| \ge c_2 n^{3/2}$.

3. (**Universal graphs**) Let $\mathcal{F}$ be a set of graphs. A graph G is said to be *universal* over $\mathcal{F}$ if every member of $\mathcal{F}$ is isomorphic to an induced subgraph of G. (Recall that an induced subgraph is determined by the choice of a subset of the vertices; no edges within that set are deleted. So for instance, every induced subgraph of a complete graph is complete.) Let $\mathcal{F}_k$ denote the set of all graphs with k vertices. Prove:

   (a) There exists a graph with $O(k^2 2^k)$ vertices that is universal over $\mathcal{F}_k$. (Hint: probabilistic method.)

(b) Give an explicit construction of a graph with $O(k^2 4^k)$ vertices that is universal over $\mathcal{F}_k$. (Hint: Paley graphs (quadratic residue graphs). Use Weil's character sum estimate.)

(c) Prove that the unique countable random graph of Problem 1 is universal over all countable graphs.

4. Prove: the chromatic number of a triangle-free graph on n vertices is $O(\sqrt{n})$.

LEFTOVERS from the apprentice problem sheets (Abért–Babai)

5. Find two infinite subsets A and B of the nonnegative integers such that every nonnegative integer can be uniquely written as the sum of an element of A and an element of B.

6. Show that for every natural number n the equation

$$\sum_{i=1}^{n} \frac{1}{a_i} = 1$$

has only a finite number of solutions in natural numbers a_i.

7. Let $\alpha_1, \ldots, \alpha_n$ be distinct real numbers. Prove that the set

$$\left\{ \frac{1}{x-\alpha_1}, \frac{1}{x-\alpha_2}, \ldots, \frac{1}{x-\alpha_n} \right\}$$

of rational functions is linearly independent over $\mathbb{R}$.

8. **(Hilbert matrix)** Let $a_1, a_2, \ldots, a_n$ be a list of n distinct numbers and $b_1, b_2, \ldots, b_n$ another list of n distinct numbers. Consider the $n \times n$ matrix $H = (h_{ij})$ with

$$h_{ij} = \frac{1}{a_i + b_j}.$$

Prove that the rows of H are linearly independent.

9. A permutation $\pi \in \mathrm{Sym}(X)$ is fixed-point-free if for all $x \in X$ we have $x^\pi \neq x$. Are there more fixed-point-free even permutations on 100 points than odd ones? (Hint: this is a determinant problem.)

10. **(Vandermonde determinant)** Let $a_1, a_2, \ldots, a_n$ be numbers. The Vandermonde matrix with generators $a_1, a_2, \ldots, a_n$ is the $n \times n$ matrix

$$V(a_1, a_2, \ldots, a_n) = \begin{bmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \ldots & a_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} \end{bmatrix}.$$

Prove:

$$\det(V(a_1, a_2, \ldots, a_n)) = \prod_{1\le i<j\le n} (a_i - a_j).$$

11. Let a, b be numbers. Verify this determinant evaluation:

$$\det \begin{bmatrix} a & b & b & \ldots & b \\ b & a & b & \ldots & b \\ b & b & a & \ldots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \ldots & a \end{bmatrix} = (a-b)^{n-1}(a+b(n-1))$$

12. Let $a_{ij} = \gcd(i, j)$ $(1 \le i, j \le n)$. Prove that for the $n \times n$ matrix $A = (a_{ij})$ we have

$$\det(A) = \varphi(1) \cdot \varphi(2) \cdot \cdots \cdot \varphi(n).$$

(φ denotes Euler's phi function.)

13. Let r be the probability that two random positive integers are relatively prime. Recall that this value is defined as a limit. Assuming the limit exists, i. e., assuming that r is well defined, give an AH-HA proof that $r = 6/\pi^2$.

14. Prove that $\sum' 1/n$ is finite, where the summation is extended over all integers which do not have the string 2006 in their decimal representation.

15. (a) Prove that there are infinitely many primes which begin with the digits 2006 (in decimal).

    (b) Prove that the sum of the reciprocals of these primes diverges.

16. A polynomial $f(x)$ is "integer-preserving" if $f(x)$ is an integer whenever x is an integer. An integer-preserving polynomial is "congruence preserving" if $f(a) \equiv f(b) \pmod m$ whenever $a \equiv b \pmod m$, for all triples of integers a, b, m. An *integral polynomial* is a polynomial with integer coefficients. Note that every integral polynomial is integer-preserving.

    (a) Find an integer-preserving polynomial which is not integral.

    (b) Find all integer-preserving polynomials.

    (c) Prove that every integral polynomial is congruence preserving.

    (d) Find a congruence preserving polynomial which is not integral.