REU 2007 · Transfinite Combinatorics · Lecture 2

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2.1 More Problems and Examples in Transfinite Combinatorics

Disjoint and Almost Disjoint Sets. Suppose $\mathscr{C}$ is a collection of non-empty disjoint subsets of $\mathbb{N}$. For each $C \in \mathscr{C}$, choose an element $n_C \in C$. Then the map $C \mapsto n_C$ is an injection of $\mathscr{F}$ into $\mathbb{N}$ which witnesses that $|\mathscr{F}| \leq |\mathbb{N}| = \aleph_0$. In other words, every collection of disjoint subsets of $\mathbb{N}$ is countable. However, by modifying the notion of "disjointness" ever so slightly, we obtain the striking result of Exercise 2.1.2 below.

Definition 2.1.1. Call sets A and B *almost disjoint* if $A \cap B$ is finite.

Exercise 2.1.2. Find continuum many almost disjoint subsets of $\mathbb{N}$. That is, construct a collection $\mathscr{C}$ of non-empty almost disjoint subsets of $\mathbb{N}$ with $|\mathscr{C}| = \mathfrak{c}$.

Exercise 2.1.3. How many disjoint circles are there in the plane? How many disjoint disks?

Definition 2.1.4. An L-*shape* is a union of two perpendicular line segments with a common endpoint. A T-*shape* is a union of two perpendicular line segments, one of which contains an endpoint of the other, but not as an endpoint.

Exercise 2.1.5. Prove that there are $\mathfrak{c}$-many disjoint L-shapes.

Exercise 2.1.6. Prove that there are at most countably many disjoint T-shapes.

The Set of Prefixes of $\mathbb{Q}$. Let $\mathscr{P}$ be the set of all prefixes of $(\mathbb{Q}, \leq)$, viewed as an order $(\mathscr{P}, \subseteq)$ under inclusion. What is the order type of this order? Naturally, $\mathscr{P}$ contains the two trivial prefixes, $\varnothing$ and $\mathbb{Q}$, which satisfy $\varnothing \subseteq P \subseteq \mathbb{Q}$ for every other prefix P. More interestingly, notice that each $r \in \mathbb{R}$ defines a prefix of $\mathbb{Q}$, namely

$$P_r = \{q \in \mathbb{Q} : q < r\},$$

and that $P_r \subseteq P_s$ if and only if $r \leq s$. However, if $r \in \mathbb{Q}$, we can also define the prefix

$$P_r^* = \{q \in \mathbb{Q} : q \leq r\},$$

and in this case we have $P_r \subset P_r^*$, and either $P_r = P$ or $P_r^* = P$ for every prefix P satisfying $P_r \subseteq P \subseteq P_r^*$. A rough picture illustrating this order type might thus look as follows:

$$\varnothing \quad \subset \quad \cdots \quad \subset \quad P_e \quad \subset \quad \cdots \quad \subset \quad P_3 \quad \subset \quad P_3^* \quad \subset \quad \cdots \quad \subset \quad \mathbb{Q}.$$

Pathological Functions. Recall the following famous functions, which serve to underscore how beginning intuition about such objects as continuous functions often fails us:

Example 2.1.7 (A bounded, nowhere continuous function.). Define $f : \mathbb{R} \to [0, 1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{otherwise} \end{cases}.$$

Example 2.1.8 (A bounded function continuous on the irrationals and discontinuous otherwise.). Define $g : \mathbb{R} \to [0, 1]$ by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ for } p \in \mathbb{Z}, q \in \mathbb{N}, (p, q) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Let us verify that this function has the desired properties. It is clear that f is discontinuous at every rational, for every rational is the limit of a sequence of irrationals, and the limit of f on any such sequence will be $0 \notin f(\mathbb{Q})$. So let x be irrational, and suppose $x_i : i \in \mathbb{N}$ is any sequence converging to x. Since $f(\mathbb{R} - \mathbb{Q}) = 0$, we may, by extracting a subsequence if necessary, assume $x_i \in \mathbb{Q}$ for all i. Writing $x_i = \frac{p_i}{q_i}$, where p_i and q_i are relatively prime integers with the latter positive, we suppose first that there exists an $N \in \mathbb{N}$ with $q_i \leq N$ for all i. Then for some $q \in \mathbb{N}$, $q_i = q$ for infinitely many i, so there is an infinite subsequence $\{x_{i_j}\}$ of $\{x_i\}$ with $q_{i_j} = q$ for all j. But this subsequence must also converge to x, which is impossible as it would mean that the p_{i_j} converge to qx, and hence that x is rational. We conclude that the q_i are unbounded, and hence that $f(x_i) = \frac{1}{q_i}$ gets arbitrarily small in the limit. In other words, $\lim_{i\to\infty} f(x_i) = 0 = f(x)$, as desired.

Exercise 2.1.9. Strengthen the last example by showing that there exists a bounded function h discontinuous on the rationals and *differentiable* on the irrationals.

Transcendental Numbers. Recall that a number $r \in \mathbb{R}$ is said to be *algebraic* if it is the root of a non-zero polynomial over $\mathbb{Z}$. For example, $\sqrt[3]{5}$, being the root of $x^3 - 5 \in \mathbb{Z}[x]$, is algebraic. A real which is not algebraic is called *transcendental.* While it is exceedingly easy to come up with one example of an algebraic number after another, it is not at all apparent how to generate examples of transcendental numbers, or, at first glance, whether there even are any. An answer to the latter question was first given by Joseph Liouville in 1844:

Theorem 2.1.10 (Liouville). *The number* $\sum_{n=0}^{\infty} \frac{1}{2^{n!}}$ *is transcendental.*

Thirty years later, Cantor was able to give a rather more insightful proof, in which he established not only that the transcendentals exist, but also that they overwhelmingly "outnumber" the algebraics in the reals.

Theorem 2.1.11 (Cantor). *The set of algebraic numbers is countable, meaning the set of transcendental numbers has power $\mathfrak{c}$.*

Proof. For each $k \in \mathbb{N}^+$, let P_k denote the set of non-zero polynomials over $\mathbb{Z}$ of degree k. Obviously, every element of P_k is completely determined by its coefficients, and hence can be uniquely reprsented by an k-tuple of integers. In other words, P_k can be regarded as a subset of $\mathbb{N}^k$, where $\mathbb{N}^1$ is defined as $\mathbb{N}$ and $\mathbb{N}^{j+1}$ for $j \geq 1$ as $\mathbb{N}^j \times \mathbb{N}$ (proper subset in fact, since we do not count any k-tuple with first coordinate 0). By induction, using the fact (proved in the last notes) that the cardinality of the product of two countable sets is countable, we have that $\mathbb{N}^k$ is countable for all k, and hence that P_k is countable for all k. Now $\mathbb{Z}[x] = \bigcup_{k\in\mathbb{N}^+} P_k$ is a countable union of countable sets, and therefore itself countable. So enumerate $\mathbb{Z}[x]$ as $\{p_0, p_1, \ldots\}$, keeping in mind that $p_i^{-1}(0)$ is finite for each i. Each algebraic number, being the root of some element of $\mathbb{Z}[x]$, can thus be assigned to $p_i^{-1}(0)$ for some i, and hence the entire set of algebraic numbers is contained in $\bigcup_{i\in\mathbb{N}} p_i^{-1}(0)$. The latter is a countable union of finite sets, hence countable, so the set of algebraic numbers is countable. $\square$

2.2 Zorn's Lemma

Definition 2.2.1. A *partially ordered set*, or *poset* for short, is a pair $(P, \leq)$, in which P is a set and $\leq$ is a binary relation on P satisfying, for all $a, b, c \in P$, the following properties:

1. $a \leq a$;
2. if $a \leq b$ and $b \leq a$, then $a = b$;
3. if $a \leq b$ and $b \leq c$, then $a \leq c$.

In other words, a poset is just an ordered set (as defined in the previous notes) in which the requirement of linearity is dropped.

Definition 2.2.2. A *chain* in a poset $(P, \leq)$ is a subset of P which forms a linear order under $\leq$.

Definition 2.2.3. 1. Let $(P, \leq)$ be a poset, and let S be a non-empty subset of P. An element $p \in P$ is said to be an *upper bound for* S if $s \leq p$ for all $s \in S$.
2. An element $p \in P$ is called *maximal* if $p = s$ whenever $p \leq s$ for some $s \in P$, and is called a *maximum* if $p \geq s$ for all $s \in S$ (note well the distinction here).

We now come to what is undeniably the most widely used application of the Axiom of Choice in mathematics. Like the Well Ordering Theorem, it is in fact equivalent to Choice.

Theorem 2.2.4 (Zorn's Lemma). *Any poset in which every chain has an upper bound, has a maximal element.*

Corollary 2.2.5. *Every vector space has a basis.*

Exercise 2.2.6. Use Zorn's Lemma to prove that in any vector space there exists a $\subseteq$-maximal linearly independent set.

Definition 2.2.7. A property of sets is called *monotone* (or *monotone increasing*, or *closed upwards*) if whenever it holds for a set S and $S \subseteq T$, then it holds for T.

Definition 2.2.8. A property of sets is called *hereditary* (or *monotone decreasing*, or *closed downards*) if whenever it holds for a set S and $S \supseteq T$, then it holds for T.

Definition 2.2.9. A property of sets if called *finitary* if it holds for a set S if and only if it holds for all finite subsets of S.

Example 2.2.10. Monotone properties include

1. being non-empty;
2. being infinite;
3. being cofinite.

Hereditary properties include

1. being empty;
2. being finite;
3. being a homogeneous set for a coloring (e.g. of pairs of natural numbers).

Corollary 2.2.11. *With respect to a hereditary, finitary property, there exists a $\subseteq$-maximal set.*

In terms of Definition 2.2.9, we can restate the result of Erdős and de Bruijn we saw last time. Zorn's Lemma lends itself to its proof.

Exercise 2.2.12 (Erdős-de Bruijn). Prove that for $k \in \mathbb{N}$, k-colorability is a finitary property. [*Hint:* Fix k and an infinite graph G in which every finite subgraph is k-colorable. Using Zorn's Lemma, take a maximal supergraph (by adding edges) which still has the property that every every finite subgraph is k-colorable. Prove that such a supergraph is complete k-partite by showing that "nonadjacent or equal" is an equivalence relation.]

2.3 König's Lemma

Consider a graph $G = (V, E)$ in which V can be written as a disjoint union (possibly infinite) of *levels* $V_0, V_1, \ldots$ such that V_0 contains a single vertex, two vertices are adjacent only if they are in consecutively-numbered levels, and each vertex has finite degree. Call such a tree a *finite-branching levelled graph*. For example, a finite-branching tree is a finite-branching levelled graph, the only difference being that in the more general definition we allow a daughter node to have two parents.

The following theorem of Dénes König is, along with Ramsey's Theorem, possibly one of the most famous result in graph theory and combinatorics. And, like Ramsey's Theorem, it has wide-ranging applications in all branches of mathematical logic.

Theorem 2.3.1 (König's Lemma). *If $G = (V, E)$ is a finite-branching levelled graph (say with $V = V_0 \cup V_1 \cup \cdots$), in which all levels are non-empty, then there exists an infinite path through G (more precisely, we can choose an element v_i in each level V_i in such a fashion that $(v_i, v_{i+1}) \in E$ for all i).*

Exercise 2.3.2. Use Zorn's Lemma to prove König's Lemma.

Exercise 2.3.3. Use König's Lemma to prove the Erdős-de Bruijn Theorem for *coutnably* infinite graphs.

Exercise 2.3.4. Recall the "tiling exercise" given in last week's exercise: given a finite set $\mathscr{T}$ of tiles, the entire plane can be tiled by the tiles in $\mathscr{T}$ if and only if the positive quadrant can. Use König's Lemma to prove the intermediate result that if all finite squares in the plane can be tiled, then so can the entire plane itself. Use this to complete the proof of the original exercise.

Finally, we present a hint for a previously-assigned exercise.

Exercise 2.3.5. Recall Sierpiński's result that $\mathfrak{c} \not\to (\aleph_1, \aleph_1)$, or in words, that a $\{\text{red}, \text{blue}\}$-coloring of $\mathbb{R}$ need not admit a homogeneous set of size $\aleph_1$. Prove first that given any $S \subseteq \mathbb{R}$, if S is well-ordered (under the standard ordering of R) then S is countable.