REU 2007 · Transfinite Combinatorics · Lecture 9

Instructor: László Babai
Scribe: Travis Schedler

August 10, 2007. Revised by instructor.
Last updated August 11, 3:40pm

Note: All $(0,1)$-measures will be assumed to be nontrivial, even when not explicitly so stated.

9.1 Axiomatizability revisited

Recall the

Theorem 9.1.1. *3-colorability is axiomatizable but not finitely axiomatizable.*

Proof. First we show axiomatizability.

First, we note that it is possible to write down an axiom to say that there are $\le n$ vertices:

$$(\exists x_1, \dots, x_n)(\forall y)(y = x_1 \vee y = x_2 \vee \cdots \vee y = x_n). \tag{1}$$

Of course, this means that it is also possible to axiomatize there being exactly n vertices (there are $\le n$ vertices but not $\le (n-1)$-vertices), but we don't need it.

Exercise 9.1.2. Given that there are $\le n$ vertices, write down an axiom to say that the graph is 3-colorable.

Now, recall that Erdős-DeBruijn says that a graph is 3-colorable iff every *finite* subgraph is 3-colorable.

We can specify axioms which test a condition on any subgraph of n elements:

B_n : $(\forall x_1, \dots, x_n)($ if all x_i are distinct then the subgraph induced on $x_1, \dots, x_n$ is 3-colorable).

By Erdős-DeBruijn, the countable set of axioms B_n $(n = 1, 2, \dots)$ defines 3-colorability.

To show that 3-colorability is not finitely axiomatizable, we show the (apparently) stronger result that *non-3-colorability* is not axiomatizable. To do this, we use ultraproducts. It suffices to construct graphs A_i which are not three-colorable whose ultraproduct $\prod A_i/\mu$ is three-colorable. Now, to do this, we could find A_i such that $\mathrm{girth}(A_i) \to \infty$. That would mean that the ultraproduct is cycle-free, i. e., a forest and therefore it is 2-colorable.

This proof uses Erdős's result that there exist finite graphs of arbitrary large girth and chromatic number. The proof was one of the great successes of the Probabilistic Method.

We can avoid using this difficult result. Indeed, it suffices to refer to the simpler result, proved in class, that there exist graphs with arbitrarily large oddgirth (oddgirth = the length of the shortest odd cycle) and large chromatic number, because then the ultraproduct will still be bipartite (it will have no odd cycles). Recall that graphs of large oddgirth and large chromatic number can be constructed by iterating the directed line graph construction of §3.1. □

Above, we used the observation that if a class of graphs (or other structures) is finitely axiomatizable then its negation must also be finitely axiomatizable. In fact, we have the following converse:

Proposition 9.1.3. *Let $\mathcal{A}$ be a class of structures over a given language. Suppose that both $\mathcal{A}$ and $\overline{\mathcal{A}}$ are axiomatizable. Then they are finitely axiomatizable.*

So in particular, if a condition (e.g., 3-colorability) is (infinitely) axiomatizable then it is not finitely axiomatizable *iff* its complement is not axiomatizable at all: what we said was apparently stronger in the above proof is in fact equivalent.

Proof. We claim that the proposition follows from the Compactness Theorem of First-Order Logic: if every finite collection of a system of axioms has a model then all of them put together have a model.

Assume $\mathcal{A}$ is axiomatizable by axioms F_i $(i \in I)$, and $\overline{\mathcal{A}}$ is axiomatizable by axioms G_j $(j \in J)$ where the index sets I, J may be infinite. Now, all F_i and all G_j together are not consistent. By the Compactness Theorem, a finite subset $F_1, \ldots, F_k$, $G_1, \ldots, G_\ell$ is already inconsistent. But now, $F_1, \ldots, F_k$ must already have been enough to define $\mathcal{A}$—otherwise, there would exist a model X of $F_1, \ldots, F_k$ which belongs to $\overline{\mathcal{A}}$: but then it must satisfy all the G_j, which would be a contradiction. $\square$

9.2 Axiomatizing properties of finite graphs

Depending on your taste, you might be really interested only in finite graphs. Now, infinite axiomatizability is irrelevant, because any property of finite graphs is certainly axiomatizable by a countable collection of axioms, each expressing exactly what it means to have that condition on a graph with a fixed number of vertices.

We would like to know whether a property is expressible using only one axiom, which need only apply to finite graphs. (Note that there are properties which become equivalent for finite things that are not equivalent for infinite ones: for example, Wedderburn's theorem that any finite division ring is in fact commutative, and thus a field. More simply, any finite integral domain must be a field.)

We have the

Claim 9.2.1. *Connectedness for finite graphs is not finitely axiomatizable.*

Proof. Suppose we are looking for a sentence φ such that if X is a *finite* graph then X satisfies φ iff X is connected. We aim to show no such sentence exists.

Suppose we have graphs that satisfy such a property φ. Their ultraproduct must also satisfy φ. (That doesn't mean that the ultraproduct is connected, just that it satisfies φ). Let us find graphs satisfying $\neg\varphi$ and show that their ultraproduct is the same, which would be a contradiction.

To do this, we simply let one collection be the n-cycles for $n = 3, 4, \ldots$, and the other collection consist of the graphs which are the disjoint union of two cycles of length n. They both have the same ultraproduct, namely the uncountable union of two-way infinite lines! $\square$

Exercise 9.2.2. Prove: bipartiteness of finite graphs is not finitely axiomatizable.

Exercise 9.2.3. Prove: 3-colorability of finite graphs is not finitely axiomatizable.

Exercise 9.2.4. Prove: planarity of finite graphs is not finitely axiomatizable.

Exercise 9.2.5. Prove: Hamiltonicity of finite *connected* graphs is not finitely axiomatizable.

What this means is that there is no sentence φ which is true for a finite connected graph if and only if the graph is Hamiltonian. Hint: complete bipartite graphs.

Exercise 9.2.6. Prove: Hamiltonicity of finite connected *planar* graphs is not finitely axiomatizable.

Hint: square grids.

9.3 The size of an ultraproduct

Here and last time, we used the following:

Claim 9.3.1. *If the size of the sets A_i goes to infinity ($i \in \mathbb{N}$), we claim that their ultraproduct has $\mathfrak{c}$ elements.*

Proof. Let us assume that $|A_i| = i$. We wish to show that there exist at least $2^{\aleph_0}$ functions on the A_i which are not pairwise equivalent ($f \sim g$ if $f(i) = g(i)$ a.a.). To find f, g such that we know this is true regardless of the measure, we need to find f, g that agree on only a finite set. So, we want $2^{\aleph_0}$ functions which pairwise agree on a finite set only. If our functions were $\mathbb{N} \to \mathbb{R}$ satisfying $0 \le f(x) \le x$, then we could accomplish this by functions $f_a(x) = ax$, for $a \in [0,1]$. However, since our functions are not real functions, we can round down: $f(x) = \lfloor ax \rfloor$. They could have finitely many overlaps, but for any two functions of this type there can be only finitely many overlaps because they will eventually diverge from each other. $\square$

Exercise 9.3.2. Replace the assumption $|A_i| = i$ with $|A_i| \to \infty$.

The assumption that $|A_i| \to \infty$ was necessary: if infinitely many $|A_i|$ were bounded by a fixed number (say, 100), then the corresponding index set can have measure 1 under some choice of the $(0,1)$-measure, and therefore the unltraproduct would also have ≤ 100 elements.

Recall that we proved that there are continuum many subsets of $\mathbb{N}$ that are pairwise almost disjoint, i. e., they have pairwise finite intersections.

We now prove that if "finite" is replaced by "bounded" in the condition then we can only have countably many subsets.

Proposition 9.3.3. *If $A_i \subseteq \mathbb{N}$ for $i \in I$ and for all $i \ne j$ we have $|A_i \cap A_j| \le 100$ then $|I| \le \aleph_0$.*

Proof. The number of ways to pick 101 things out of $\aleph_0$ is still $\aleph_0$. No two of the A_i share 101 elements, so the number of sets A_i can be at most the number of ways of picking 101 elements out of countably many. $\square$

9.4 The Random Graph revisited

Exercise 9.4.1. Suppose that φ is a sentence, and p_n is the probability that a graph on n vertices satisfies φ. Prove that $\lim_{n\to\infty} p_n$ is either 0 or 1.

Proof. To solve the above exercise, we claim that $\lim_{n\to\infty} p_n = 0$ iff φ is false for the countable Random Graph (from last time), while the limit is 1 if φ is true for the countable Random Graph.

Recall that the Random Graph is the unique countable model of the axioms $Ax_{n,\varepsilon}$ from last time (specifying that, for all sequences of n distinct vertices, there exists a vertex satisfying the adjacency relation specified by the function $\varepsilon : n \to 2$ with them).

So either φ or $\neg\varphi$ is inconsistent with the axioms $Ax_{n,\varepsilon}$; and so by Compactness, it is inconsistent with a finite number of them. Let us assume, for example, that φ is inconsistent with a finite subset $\mathcal{F}$ of the axioms $Ax_{n,\varepsilon}$. Now show that any finite number of the $Ax_{n,\varepsilon}$ are satisfied by almost all finite graphs. In particular, almost all finite graphs satisfy all axioms in $\mathcal{F}$ and therefore they do not saisfy φ, so $p_n \to 0$. The same argument applied to $\neg\varphi$ completes the other case. □

9.5 Ramsey's Theorem revisited

We give another proof of Ramsey's Theorem $\aleph_0 \to (\aleph_0, \aleph_0)$.

Proof. Consider a finitely additive $(0, 1)$-measure on $\mathbb{N}$, the vertex set of our complete graph of which the edges have been colored red/blue.

Then one of the following holds for almost all vertices v:

(r) almost all vertices w are connected to v with a red edge;

(b) almost all vertices w are connected to v with a blue edge.

WLOG assume (r) holds for almost all v. Delete all vertices for which it does not hold; this deletes only a zero-set, so now (r) holds for all vertices.

Now define the sequence v_i of vertices inductively as follows.

Let v_i be a vertex that is connected to all v_j $(j < i)$ by a red edge. Given $\{v_j : j < i\}$, we can choose v_i from a set of measure 1, so such v_i exists. Now the sequence $(v_i : i \in \mathbb{N})$ induces a countable complete graph in red. □

Let us now show that $\aleph_0 \to (\aleph_0, \aleph_0)_3$. This means the following:

Definition 9.5.1. Let $\binom{A}{r}$ = the set of r-subsets of A. Then, $n \to (k, \ell)_r$ means that, if $\binom{n}{r} = R \dot\cup B$, then $\exists A_1 \subseteq n, |A_1| = k, \binom{A_1}{r} \subseteq R$, or $\exists A_2 \subseteq n, |A_2| = \ell, \binom{A_2}{r} \subseteq B$.

The previously used notation $n \to (k, \ell)$ now means $n \to (k, \ell)_2$ (we colored the pairs of vertices).

Theorem 9.5.2. ***(Ramsey's Theorem (unabridged))***

1. *For all $r, t, k_1, \dots, k_t$, there exists N such that $N \to (k_1, \dots, k_t)_r$. (All parameters in this statement are finite.)*
2. *$\aleph_0 \to (\aleph_0, \dots, \aleph_0)_r$ (t times).*

Exercise 9.5.3. Show that 2 implies 1.

Exercise* 9.5.4. Give an effective bound on N.

Exercise 9.5.5. Prove 2 using ultrafilters.

We see that ultrafilters really are a machine for producing theorems!

9.6 Ordinal Ramsey Theory revisited

Proposition 9.6.1. *If* $\alpha < \omega_1$ *then* $\alpha \nrightarrow (\omega, \omega + 1)$*.*

Proof. Given vertices labeled by α, we find a coloring (of the complete graph on those vertices) by red and blue colors so that any complete red subgraph has ordinal type $\leq \omega$ and any complete blue subgraph is finite, which solves the problem. The idea is to form a comparison graph, as was done in Solution 7.3.2.

Suppose that $(\alpha, \leq)$ is α equipped with its defining ordering. Since α is countable, there is a bijection to ω, and let $(\alpha, \prec)$ be the induced ordering of type ω. is ordered in ω. Form a comparison graph: we place a red line $i \sim j$ if the two orders agree on (i, j); otherwise we place a blue arrow. Any complete red subgraph must have vertex set injecting (preserving order) into ω, since the orders $\leq, \prec$ agree. Hence, the vertices must have ordinal type $\leq \omega$. On the other hand, any red subgraph must be finite, otherwise there would be an infinite decreasing subchain of $(\alpha, \leq)$, which is impossible because $\leq$ is well-ordered. □

9.7 Measurable implies strongly inaccessible

Recall:

Definition 9.7.1. A cardinal κ is **measurable** if there exists a nontrivial, $< \kappa$-additive $(0, 1)$-measure, i. e., $|I| < \kappa \Rightarrow \mu(\dot{\bigcup}_{i \in I} A_i) = \sum_{i \in I} \mu(A_i)$.

Note that $\aleph_0$ is measurable. We shall prove that the next measurable cardinal must be enormous.

Exercise 9.7.2. If κ is measurable then $\kappa \to (\kappa, \kappa)$.

In fact, a much stronger statement holds:

Exercise 9.7.3. If κ is measurable then $\kappa \to (\kappa, \ldots, \kappa)_r$ (t times), for all $r < \omega$ and $t < \kappa$.

Exercise 9.7.4. $2^{\aleph_0}$ has no $\aleph_0$-additive $(0, 1)$-measure.

Solution. Write a binary sequence

$$0101101110001\ldots \tag{2}$$

such that, at each place, we pick the digit that occurs there with probability 1. Then, the number above is a single number which is the intersection of countably many sets of measure 1, and hence has measure 1.

Exercise 9.7.5. Generalize this to show that if λ is an infinite cardinal then on 2^λ there is no λ-additive $(0, 1)$-measure.

From this it follows that if κ is measurable and $\lambda < \kappa$ then $2^\lambda < \kappa$.

Definition 9.7.6. An infinite cardinal m is **singular** if it is the sum of fewer, smaller cardinals:

$$m = \sum_{i \in I} n_i, \tag{3}$$

where $n_i < m$ and $|I| < m$. Otherwise, m is called **regular**.

Note that $\aleph_{\alpha+1}$ is regular.

For an example of a singular cardinal, $\aleph_\omega = \sum_{n<\omega} \aleph_n$.

Exercise 9.7.7. (a) $\aleph_{\omega_1}$ is singular.
(b) If α is a limit ordinal and $\aleph_\alpha$ is regular then $\alpha = \omega_\alpha$.

Definition 9.7.8. m is **weakly inaccessible** if it is a regular limit cardinal.

Definition 9.7.9. m is **strongly inaccessible** if

(i) it is regular, and

(ii) $\lambda < m$ implies $2^\lambda < m$.

Theorem 9.7.10. *If κ is measurable then κ is strongly inaccessible.*

Proof. Any subset of κ of size $< \kappa$ must have measure zero by $< \kappa$-additivity; so if κ were a sum of fewer smaller cardinals then those smaller cardinals would have measure zero; but then their union, which is all of κ, would also have measure zero, contradicting the definition of measure. So κ must be regular. For the second property, see Exercise 9.7.5. $\square$

The smallest strongly inaccessible cardinal $> \omega$ cannot be measurable; in fact, if s_α denotes the α-th strongly inaccessible cardinal and $\kappa > \omega$ is measurable then $\kappa = s_\kappa$.

Ultraproducts over measurable cardinals have been used to great effect. In particular, by studying the κ-th ultrapower of a model of ZFC (with respect to a $< \kappa$-additive $(0, 1)$-measure), Gaifman and Rowbottom proved in 1964 that the existence of a measurable cardinal $> \omega$ radically affects the landscape of sets of low cardinality; specifically, in that case, ω has only countably many "constructible" subsets (although, as Gödel had shown, it is consistent with ZFC that *every* set is constructible).

Comments on notation and terminology. In the literature, "λ-additivity" of a measure means "additive over any set of **fewer than** λ sets." So what we called $< \lambda$-additive is in fact called λ-additive. For instance, a finitely additive measure is "ω-additive" and a countably additive or "σ-additive" measure is "ω_1-additive." While this convention is more natural in the context of set theory, it may be somewhat confusing to the novice, which is the reason why I deviated from it.

Another deviation is that I included $\aleph_0$ among the measurable cardinals whereas in the literature, measurable cardinals are typically defined to be uncountable. Excluding $\aleph_0$ from among the measurable cardinals is not a natural convention and its sole reason is to avoid having to say over and over that we are talking about *uncountable* measurable cardinals. I chose to deviate from this convention to emphasize the analogy between $\aleph_0$ and the measurable cardinals, exemplified by the measures themselves, and their consequences, including Ramsey properties and ultraproducts.

Acknowledgments The instructor wishes to thank his friend and mentor, Lajos Pósa, for the weekly math club Pósa, then an undergraduate freshman at Eötvös University, Budapest, held in the distant past at Fazekas High School. The instructor, then an 11th grader at Fazekas High, learned the love of creative problem solving and the transfinite universe as well as a considerable portion of the actual material presented in the past three weeks, during those math club sessions.

The instructor expresses his gratitude to the scribes for their excellent and timely work in drafting these lecture notes which greatly enhance the value of the course.