REU APPRENTICE CLASS #10

INSTRUCTOR: LÁSZLÓ BABAI
SCRIBE: MICHAEL SMITH

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1. Determinant Expansion

For any $A \in M_n(F)$, define A_{ij} to be the $(n-1) \times (n-1)$ submatrix obtained from A by removing row i and column j.

Theorem 1.1 (Expansion by i^{th} row or cofactor expansion)**.** *For any $A \in M_n(F)$ and $1 \leq i \leq n$,*

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Problem 91. Prove this theorem.

Observation 1.2. If $\ell \neq i$, the "skew expansion"

$$\sum_{j=1}^{n} (-1)^{i+j} a_{\ell j} \det(A_{ij})$$

is zero.

Definition 1.3. The number

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

is called the (i, j)-cofactor of A.

Theorem 1.1 can be stated as

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}.$$

Definition 1.4. The *adjoint matrix* of A is the transpose of the matrix of cofactors of A:

$$\operatorname{adj}(A) = (C_{ij})^T$$

Theorem 1.5. *For all square matrices A we have $A \cdot \operatorname{adj}(A) = \det(A) \cdot I$. In particular, if $\det A \neq 0$, then*

$$A \cdot \left(\frac{1}{\det A} \operatorname{adj}(A) \right) = I_n,$$

i.e.

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A).$$

Exercise 1.6. If there is a b such that the equation $Ax = b$ has a unique solution, then A is nonsingular.

Problem 92. If $A, B \in M_n(F)$, then $\det(AB) = \det(A) \cdot \det(B)$.

For two vectors (a, b) and (c, d) in $\mathbb{R}^2$, the area of the parallelogram they define is

$$\left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|$$

and the sign of this determinant indicates whether the second vector is clockwise or counterclockwise of the first.

For three vectors in $\mathbb{R}^3$, the absolute value of the corresponding determinant gives the volume of the parallelepiped they define, and the sign of this determinant indicates the handedness of the system.

Problem 93. The area of a parallelogram in $\mathbb{R}^3$ defined by two integer vectors is the square root of an integer.

Problem 94. Find infinitely many positive integers that cannot be written as $a^2+b^2+c^2$ for integers a, b, c.

Theorem 1.7. *If $F \subseteq H$ is a field extension and $A \in F^{k\times n}$, then $\mathrm{rk}_F(A) = \mathrm{rk}_H(A)$.*

Observation 1.8. If $A \in \mathbb{Z}^{k\times n}$, then we can obtain a corresponding matrix in $F^{k\times n}$ for any field F, and the rank of this matrix depends only on the characteristic of F.

Notation 1.9. If $A \in \mathbb{Z}^{k\times n}$, write $\mathrm{rk}_p(A)$ for $\mathrm{rk}_{\mathbb{F}_p}(A)$, and $\mathrm{rk}_0(A)$ for $\mathrm{rk}_{\mathbb{Q}}(A)$.

Problem 95.

- Find a matrix A whose entries are all 0 or 1 such that $\mathrm{rk}_2(A) \neq \mathrm{rk}_0(A)$.
- Prove that if A is an integer matrix, then $\mathrm{rk}_p(A) \leq \mathrm{rk}_0(A)$.

2. Dot Products

In this section, F is an arbitrary field.

Definition 2.1. In F^n, the *standard dot product* of two vectors $a = (\alpha_1, \dots, \alpha_n)^T$ and $b = (\beta_1, \dots, \beta_n)^T$ is given by

$$a \cdot b = \sum_{i=1}^{n} \alpha_i \beta_i.$$

In other words, $a \cdot b = a^T b$.

Definition 2.2. Two vectors $a, b \in F^n$ are *orthogonal* (or *perpendicular*), written $a \perp b$, if $a \cdot b = 0$. A vector $a \in F^n$ is orthogonal to a set $S \subseteq F^n$, written as $a \perp S$, if $(\forall s \in S)(a \perp s)$. Two sets $S, T \subseteq F^n$ are orthogonal, written as $S \perp T$, if $(\forall s \in S)(\forall t \in T)(s \perp t)$.

Definition 2.3. If S is a subset of F^n, define the set $S^\perp$ (pron. "S-perp") as

$$S^\perp = \{u \in F^n \mid u \perp S\}$$

Exercise 2.4. $S^\perp$ is a subspace of F^n.

Exercise 2.5. (a) $u \perp S$ if and only if $u \perp \mathrm{Span}(S)$. (b) $S^\perp = (\mathrm{Span}(S))^\perp$.

Exercise 2.6. $S \subseteq (S^\perp)^\perp$.

Problem 96. $\dim S^\perp = n - \mathrm{rk}(S)$.

Problem 97. If $U \leq F^n$ then $\dim U + \dim(U^\perp) = n$.

Problem 98. If $U \leq F^n$ then $(U^\perp)^\perp = U$.

Problem 99. If $U \leq \mathbb{R}^n$, then $U \cap U^\perp = \{0\}$.

Observation 2.7. If $U = U^\perp$, then $\dim U = n/2$.

Problem 100. For all even n, and for $F = \mathbb{C}, \mathbb{F}_5$, and $\mathbb{F}_p$ for infinitely many other primes p (which ones?), find an $(n/2)$-dimensional subspace U of F^n such that $U = U^\perp$.

Definition 2.8. A vector $u \in F^n$ is *isotropic* if $u \perp u$. A subspace U is *totally isotropic* if $U \perp U$, i. e., if $U \subseteq U^\perp$.

Problem 101. (a) If $U \leq F^n$ is totally isotropic, then $\dim U \leq n/2$. (b) Use this to prove that in Eventown, the number of clubs is at most $2^{\lfloor n/2 \rfloor}$. (Recall that in Eventown, every club has an even number of members and every pair of clubs shares an even number of members.)