REU APPRENTICE CLASS #12

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Problem Session

(17 students present.)
Today we discussed problems 75, 88, 89, 90, 93, 94, 96, 97, 99, 100, 101, 109. Some remarks:

- **Problem 75**: Nathan proposed that $F = \mathbb{Q}[\sqrt[10]{2}]$ satisfies $[F : \mathbb{Q}] = 10$. Write $\alpha = \sqrt[10]{2}$. We need to show $1, \alpha, \alpha^2, ..., \alpha^9$ are linearly independent over $\mathbb{Q}$. It suffices to show that $f(x) = x^{10} - 2$ is irreducible over $\mathbb{Q}$.
- **Problem 88**: Remember that computing determinants is easier when there are a lot of zeroes. When there are a lot of equal entries, we can get a matrix with a lot of zeroes by doing row operations. Indeed: This determinant is easily computed by putting it into upper-triangular form using row operations. It is also useful to remember that if every entry of a row (or column) has the same factor, then that factor may be pulled out of the determinant. The final answer we found was $(a + (n - 1)b)(a - b)^{n-1}$.
- **Problem 89**: Observe that the determinant of the Vandermonde matrix vanishes if we have $x_i = x_j$ for any $i \neq j$. Consequently, we might guess the determinant to be
$$D = \prod_{i<j}(x_j - x_i).$$
This is indeed the correct answer. Two ways of proving this: 1. Observe that D is a polynomial of the correct degree and correct leading coefficient. By uniqueness of factorization into irreducibles of multivariate polynomials [a fact we haven't discussed], D must equal the determinant $\det V(x_1, ..., x_n)$. 2. Use Gaussian elimination to find
$$V_n(x_1, ..., x_n) = \prod_{j=2}^{n}(x_j - x_1)V_{n-1}(x_2, ..., x_n).$$
- **Problem 90**: Zach proposed the answer $D_n = F_{n+1}$. To prove this, we must show D_n satisfies the Fibonacci recurrence $D_n = D_{n-1} + D_{n-2}$. **Homework: Prove this.**
- **Problem 93**: David solved this by observing that $|x \times y| = \text{area}$. See Problem **??** below.
- **Problem 94**: Peter proposed that if $n \equiv -1 \pmod 8$, then $n \neq a^2 + b^2 + c^2$ for integers a, b, c. **Homework: Prove this.**
- **Problem 96**: Observe that $S^\perp$ is a subspace and $S^\perp = (\text{span}(S))^\perp$. Take a basis $v_1, ..., v_k$ for $\text{span}(S)$; then $x \in S^\perp$ if and only if $x \cdot v_i = 0$ for $i = 1, ..., k$. This gives a system of k independent linear homogeneous equations. The dimension of the solution space is $n - k$ (by the rank-nullity theorem). Thus, $\dim S^\perp = n - \text{rk}(S)$. *Let us emphasize this point:* if $U \leq F^n$ and B is a basis of U then $U^\perp = B^\perp$.
- **Problem 97**: This is a special case of Problem 96. This implies a totally isotropic subspace U (i. e., $U \perp U$, i. e., $U \subseteq U^\perp$) satisfies $\dim U \leq \lfloor \frac{n}{2} \rfloor$.
- **Problem 99**: Zihao solved this problem by observing that for $u = (\alpha_1, ..., \alpha_n)$, we have $u \cdot u = \sum_{i=1}^n \alpha_i^2 = 0$ if and only if $u = 0$ (in $\mathbb{R}^n$). Note this is not true over other fields: For example, we have
$$\begin{pmatrix}1\\2\end{pmatrix} \cdot \begin{pmatrix}1\\2\end{pmatrix} \equiv 1^2 + 2^2 \equiv 0 \pmod 5$$

in $\mathbb{F}_5^2$, and

$$\begin{pmatrix}1\\i\end{pmatrix}\cdot\begin{pmatrix}1\\i\end{pmatrix}=0$$

in $\mathbb{C}^2$.

- **Problem 100**: The trick is to use vectors like

$$\begin{pmatrix}1\\a\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\a\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\1\\a\end{pmatrix},$$

where a is a square root of -1 in F. See remarks on Problem 99 above and look at Problem 111 below.

- **Problem 101**: Let v_A denote the characteristic vector (membership vector) for the set $A \subseteq \{1,\dots,n\}$ (e.g., for $A=\{1,3,4\}\subseteq\{1,\dots,5\}$ we have $v_A=(1,0,1,1,0)$). Then $v_A\cdot v_B=|A\cap B|$. In particular, $v_A\cdot v_A=|A|$. Let v_i be the membership vectors for clubs in Eventown. Since all the clubs have an even number of members and all pairs of clubs share an even number of members, we find $v_i\cdot v_j=0$ over $\mathbb{F}_2$ for all i,j. So if S is a set of membership vectors, then the corresponding sets satisfy the Eventown conditions if and only if $S\perp S$. It follows that the membership vectors of a maximal Eventown club system form a totally isotropic subspace of $\mathbb{F}_2^n$. Now look at Problem 97 above.

- **Problem 109**: David presented a solution. Recall that the characteristic polynomial is

$$f_A(t)=\det(tI-A)=(t-\lambda_1)\cdots(t-\lambda_n),$$

where λ_i are the eigenvalues.

($\Rightarrow$) If A is diagonal, it is trivial that the algebraic and geometric multiplicities are the same. Now observe that if A,B are similar matrices then every λ has the same algebraic multiplicity for A as for B (because A and B have the same characteristic polynomial); and λ has the same geometric multiplicity for the two matrices as well, because the geometric multiplicity of λ is $n-\mathrm{rk}(\lambda I-A)$ for A and $n-\mathrm{rk}(\lambda I-B)$ for B, and $\lambda I-A\sim\lambda I-B$ (why?). So the equality of the algebraic and geometric multiplicities carries over to diagonalizable matrices.

($\Leftarrow$) Recall that A is diagonalizable if and only if A has an eigenbasis. Write the characteristic polynomial as

$$f_A(t)=\prod_{i=1}^{k}(t-\lambda_i)^{k_i},$$

where $\lambda_1,...,\lambda_k$ are the distinct eigenvalues and k_i is the algebraic multiplicity of λ_i. Then $\sum_{i=1}^k k_i=n$. This implies that $\sum_{i=1}^k \text{geom.mult.}(\lambda_i)=n$ by the hypothesis. Now pick a basis for each eigenspace U_{λ_i} and combine all these bases. We claim that the union of the eigenbses of each eigenspace forms an eigenbasis for A. We have the right number of vectors, so all we need to verify is that they are linearly independent. To complete the verification, solve Problem 113 below.

Some New Problems

Problem 110. If $a_1,...,a_k\in\mathbb{Z}^n$, then $\text{volume}_k(\text{para.}(a_1,...,a_k)=\sqrt{\text{integer}}$, where

$$\text{para.}(a_1,...,a_k)=\left\{\sum_{i=1}^{n}\alpha_i a_i \mid 0\le\alpha_i\le 1\right\}$$

is the parallelepiped spanned by $a_1,\dots,a_k$.

Problem 111. For what primes p does there exist $\sqrt{-1}$ in $\mathbb{F}_p$?

Problem 112. If U is a totally isotropic subspace of $\mathbb{F}_2^n$ and $\dim U<\lfloor\frac{n}{2}\rfloor$, then U is not *maximal*, that is, there exists a totally isotropic subspace $U'\le\mathbb{F}_2^n$ such that $U'\supsetneq U$.

Problem 113. If $v_1, ..., v_k$ are eigenvectors of A to **distinct** eigenvalues ($v_i \neq 0$, $Av_i = \lambda_i v_i$, $\lambda_i \neq \lambda_j$ for $i \neq j$), then the v_i are linearly independent.