REU APPRENTICE CLASS #14

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Friday, July 15, 2011

1. Algebraically closed fields

Definition 1.1. The field F is *algebraically closed* if any of the following equivalent conditions hold:

- For all polynomials $f \in F[x]$ with $\deg f \geq 1$, there is an $\alpha \in F$ with $f(\alpha) = 0$.
- Every nonzero polynomial over F factors into linear factors.
- The only irreducible polynomials over F are linear.

Theorem 1.2. *If F is a field, there is an extension $H \supseteq F$ of F such that H is algebraically closed.*

Problem 131. Let $F \subseteq H$ be a field extension, and let $\mathrm{Alg}_F(H) = \{\alpha \in H \mid \alpha \text{ is algebraic over } F\}$. Then

- $\mathrm{Alg}_F(H)$ is a subfield of H.
- If H is algebraically closed, then $\mathrm{Alg}_F(H)$ is algebraically closed.
- In this case, $\mathrm{Alg}_F(H)$ is the smallest algebraically closed field containing F. Call this field the algebraic closure of F, and denote it by $\overline{F}$.
- Show that $\overline{F}$ is unique up to an isomorphism fixing F.

Problem 132. If F is countable, then $\overline{F}$ is countable.

Definition 1.3. α is *transcendental* over F if α is not algebraic over F.

Problem 133 (Liouville)**.** Show that

$$\sum_{n=0}^{\infty} \frac{1}{2^{n!}}$$

is transcendental.

2. Irreducible polynomials

Definition 2.1. A polynomial $f \in \mathbb{Z}[x]$, $f(x) = a_0 + a_1x + \cdots + a_nx^n$ is *primitive* if

$$\gcd(a_0, \ldots, a_n) = 1.$$

Problem 134 (Gauss Lemma #1)**.** Show that the product of primitive polynomials is primitive.

Definition 2.2. Two factorizations $f = g_1 \ldots g_k = h_1 \ldots h_\ell$ are *equivalent* if $k = \ell$ and there exist $\alpha_1, \ldots, \alpha_k \in \mathbb{Q}$ such that $h_i = \alpha_i g_i$.

Theorem 2.3 (Gauss Lemma #2)**.** *Let $f \in \mathbb{Z}[x]$ and $f = g_1 \ldots g_k$ be a factorization into polynomials $g_i \in \mathbb{Q}[x]$. Then there is an equivalent factorization $f = h_1 \ldots h_k$ into polynomials $h_i \in \mathbb{Z}[x]$.*

Problem 135. Prove Theorem 2.3.

Theorem 2.4 (Schönemann-Eisenstein criterion)**.** *Let $f \in \mathbb{Z}[x]$, $f = a_0 + a_1x + \cdots + a_nx^n$. Suppose there is a prime p such that $p \nmid a_n$, $p \mid a_0, \ldots, a_{n-1}$, and $p^2 \nmid a_0$. Then f is irreducible over $\mathbb{Q}$.*

Problem 136. Prove Theorem 2.4

Problem 137. Prove that $x^{10} - 8$ is irreducible.

Definition 2.5. The n^{th} *cyclotomic polynomial* is the polynomial

$$\Phi_n(x) = \prod_{\omega \text{ a primitive } n^{\text{th}} \text{ root of unity}} (x - \omega).$$

Problem 138.

- Calculate $\Phi_n(x)$ for $n \leq 10$.
- Prove that $\Phi_n(x)$ has integer coefficients.
- * Prove that $\Phi_n(x)$ is irreducible.

Theorem 2.6. *$\Phi_p(x)$ is irreducible.*

Problem 139. Let $A \in M_n(\mathbb{Q})$. If $f_A(t)$ is irreducible over $\mathbb{Q}$, then A is diagonalizable over $\mathbb{C}$.

3. The Cayley-Hamilton Theorem

Definition 3.1. Let $f \in F[x]$, $A \in M_n(F)$. If $f(x) = a_0 + a_1 x + \cdots + a_k x^k$, then we define $f(A) = a_0 I + a_1 A + \cdots + a_k A^k$.

Theorem 3.2 (Cayley-Hamilton)**.** *For any $A \in M_n(A)$, $f_A(A) = 0$.*

Notation 3.3.

$$\operatorname{diag}(r_1, \ldots, r_n) = \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix}$$

Observation 3.4. Let $*$ denote either addition or multiplication. Then $\big(\operatorname{diag}(r_1, \ldots, r_n)\big) * \big(\operatorname{diag}(s_1, \ldots, s_n)\big) == \operatorname{diag}\big(r_1 * s_1, \ldots, r_n * s_n\big)$. In particular, diagonal matrices commute.

Corollary 3.5. *Let $f \in F[x]$. Then $f\big(\operatorname{diag}(r_1, \ldots, r_n)\big) = \operatorname{diag}\big(f(r_1), \ldots, f(r_n)\big)$.*

Corollary 3.6. *The Cayley-Hamilton Theorem holds for diagonal matrices.*

Observation 3.7. $f(S^{-1}AS) = S^{-1}f(A)S$, and hence if $A \sim B$ then $f(A) \sim f(B)$. Additionally, if $A \sim B$, $f_A(t) = f_B(t)$.

Corollary 3.8. *The Cayley-Hamilton theorem holds for all diagonalizable matrices.*

Exercise 3.9. If $A_j \to A$ and $B_j \to B$, then $A_j + B_j \to A + B$ and $A_j B_j \to AB$.

Exercise 3.10. If $A_j \to A$, then $f_{A_j} \to f_A$.

Exercise 3.11. If $f_j \to f$ and $A_j \to A$, then $f_j(A_j) \to f(A)$.

Corollary 3.12. *The set of matrices for which the Cayley-Hamilton theorem holds is closed (in the topological or analytic sense, i.e. closed under limits).*

So to prove the Cayley-Hamilton Theorem for all complex matrices, it suffices to prove this:

Theorem 3.13. *Diagonalizable matrices are dense in $M_n(\mathbb{C})$, i.e. for all $A \in M_n(\mathbb{C})$ there is a sequence A_j in $M_n(\mathbb{C})$ such that $A_j \to A$ and for all j, A_j is diagonalizable.*

Actually something much stronger is true:

Problem 140. Almost all matrices in $M_n(\mathbb{C})$ are diagonalizable. That is, the set of non-diagonalizable matrices has Lebesgue measure zero.

We shall not use this result to prove Theorem 3.13. Instead, we will use the following result which is important in its own right.

Theorem 3.14. *For all $A \in M_n(\mathbb{C})$ there is an invertible $S \in M_n(\mathbb{C})$ such that $S^{-1}AS$ is upper triangular.*

Problem 141. Prove Theorem 3.14.

Lemma 3.15. *If $A \in M_n(\mathbb{C})$ and all eigenvalues of A are distinct, then A is diagonalizable.*

Observe that Lemma 3.15 combined with Theorem 3.14 prove that the diagonalizable matrices are dense (Theorem 3.13) and therefore completes the proof of the Cayley-Hamilton Theorem for complex matrices (except that we haven't proved Theorem 3.14 yet).

Problem 142. Show that if the Cayley-Hamilton theorem holds over the integers, it holds over every commutative ring with an identity element.

Problem 143.

- Show that if $A, B \in M_n(\mathbb{R})$, then $AB - BA \neq I$.
- Show the same is true over all fields of characteristic zero.
- Find a field for which this is not true.
- Find two linear transformations A and B of $\mathbb{R}[x]$ such that $AB - BA = I$. (In class I mistakenly suggested trying the space $\ell^2(\mathbb{N})$; I thank Michael for pointing out that this choice does not work.)