REU APPRENTICE CLASS #15

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1. Nonnegative matrices, Directed Graphs, Perron-Frobenius Theory

Problem 144. Let A be an $n \times n$ matrix over a field F. Suppose that u is a right eigenvector, that is $Au = \lambda u$, and v is a left eigenvector, that is $v^t A = \mu v$. If $\lambda \neq \mu$, then show that $u \perp v$.

Problem 145. Let A be an $n \times n$ matrix over $\mathbb{R}$. Let $A = (a_{ij})$.

(a) Suppose that for all i, j, we have $a_{ij} > 0$. Show that there exists an eigenvector with all positive coordinates.

(b) Now suppose that for all i, j, we have $a_{ij} \geq 0$. Construct a directed graph (digraph) with vertex set $\{1, \ldots, n\}$ as follows. Put an edge (arrow) $i \to j$ if $a_{ij} \neq 0$. Suppose that the associated digraph is strongly connected. That is, for all i, j there is a (directed) path from i to j. Show that there is an all-positive eigenvector.

(c) Suppose that for all i, j, we have $a_{ij} \geq 0$. Show that there exists a non-negative eigenvector.

(d) Show that parts (b) and (c) imply that every finite Markov Chain has a stationary distribution.

(e) If a digraph is strongly connected, then it follows that such a stationary distribution is unique.

(f) Construct a Markov Chain with non-unique stationary distribution.

These are elements of the Perron-Frobenius theory of non-negative matrices, which are key to the theory of finite Markov Chains.

Recall that a *walk* is a sequence of directed edges such that each edge ends where the next one begins. (Nodes and edges may be repeated along the walk.) A *path* is a walk without repeated nodes.

Definition 1.1. The *period* of a vertex v in a directed graph is the gcd of the lengths of all closed walks from the vertex v.

Exercise 1.2. Find a digraph that is weakly connected but not strongly connected.

Definition 1.3. We say that a vertex y is *reachable* from a vertex x if there exists a walk from x to y.

Exercise 1.4. Show that if y is reachable from x, then there is a path from x to y.

Definition 1.5. We say (only in this class) that x and y are *equivalent* if they are mutually reachable from each other.

Exercise 1.6. Prove that mutual reachability is an equivalence relation. That is, show that it is reflexive, symmetric, and transitive.

The equivalence classes of this relation are called *strong components*.

Problem 146. If two vertices of a digraph are in the same strong component, then they have the same period.

Definition 1.7. If X is a strongly connected digraph, then its *period* is the period of any vertex v (since the periods of all vertices are equal).

Recall that a closed path in a graph is called a *cycle*.

Problem 147. Assume that X is strongly connected. Then the period of X is the gcd of the lengths of all the cycles in X.

Problem 148. All closed walks in a graph have lengths divisible by some natural number d if and only if the vertices can be grouped in d blocks around a circle, such that edges only go from one block to the next along the circle.

Problem 149. Let A be an $n \times n$ matrix over $\mathbb{C}$. Let X_A be the associated digraph on the vertex set $\{1, \ldots, n\}$. That is, there is an edge $i \to j$ if $a_{ij} \neq 0$. Suppose that X_A is strongly connected with period h. Let ω be an h-th root of unity. Show that if λ is an eigenvalue of A, then so is $\lambda\omega$.

2. Undirected graphs, Hamiltonicity, automorphisms

Definition 2.1. A *Hamilton cycle* in an undirected graph is a cycle that passes through every vertex of the graph (exactly once). A graph is called *Hamiltonian* if it has a Hamilton cycle.

In class, we drew some examples of graphs that are and are not Hamiltonian.

Definition 2.2. A connected graph with no cycles is called a *tree*.

Definition 2.3. A *cut vertex* of a connected graph is a vertex such that if it is removed, then the graph becomes disconnected.

Definition 2.4. If a connected graph has more than two vertices but has no cut-vertex, then it is called *2-connected*.

We can construct the following graph X that is 2-connected but not Hamiltonian. Let X have five vertices, namely x, y, a, b, c. Connect x to a, b, c and y to a, b, c. This graph is 2-connected. However, it cannot be Hamiltonian. This is because removing the vertices x, y breaks the graph into three connected components, whereas removing k vertices in a Hamiltonian graph can yield at most k connected components.

Definition 2.5. An undirected graph X is called *tough* if for every $k \in \mathbb{N}$ if k vertices are removed from X, then there are at most k connected components.

Observation 2.6. If a graph is Hamiltonian, then it is tough.

Problem 150. Show that not every tough graph is Hamiltonian. (Only consider graphs with at least three vertices.)

Definition 2.7. An *isomorphism* of two graphs X and Y is a bijection between the vertex sets of X and Y, which preserves adjacency. An *automorphism* or *symmetry* of a graph X is an isomorphism from X to itself.

For example, let X be a graph consisting of five vertices and five edges arranged in a cycle. Then the symmetries of X consist of five rotations and five reflections, and they form the group D_5, called the dihedral group of degree 5.

Consider the Petersen graph, which consists of an outer ring of five vertices arranged as a pentagon, an inner ring of five vertices arranged as a star, and with corresponding vertices in the inner and outer rings connected. Let us find some symmetries (automorphisms) of the Petersen graph.

Problem 151. Does Petersen's graph have an automorphism that interchanges the outer five and the inner five vertices?

Definition 2.8. The *girth* of a graph X is the length of the shortest cycle in the graph. If there is no cycle, then the girth is said to be ∞.

For example, the Petersen graph has girth 5.

Problem 152. In class, we drew another 3-regular graph of girth 5 with 10 vertices. Is this graph isomorphic to the Petersen graph?

Problem 153. Show that Petersen's graph is not Hamiltonian.

Definition 2.9. A graph X is called *vertex-transitive* if for any two vertices x and y, there exists an automorphism π of X such that $\pi(x) = y$.

Problem 154. (a) Show that Petersen's graph is tough.
(b) Show that Petersen's graph is vertex-transitive.
(c) ($*$) Show that every connected vertex-transitive graph is tough.

Open question. Are there infinitely many connected vertex-transitive non-Hamiltonian graphs? Only four examples are known.

Problem 155. For what values of $k, \ell \in \mathbb{N}$ is the $k \times \ell$ grid Hamiltonian?

Problem 156. Consider a $3 \times 3 \times 3$ grid of cubes of cheese. Suppose that a mouse wants to eat all cheese cubes one at a time such that any two consecutive cubes share a common face. Suppose that the mouse wants to eat the centre cube last. Show that this is not possible.

Problem 157.

(a) The number of automorphisms of Petersen's graph is 120.
(b) Is the automorphism group of Petersen's graph isomorphic to S_5?
(c) Show that the order of the automorphism group of the dodecahedron is 120.
(d) Show that the automorphism group of the dodecahedron is not isomorphic to S_5.
(e) Show that half of the automorphisms of the dodecahedron form a subgroup isomorphic to a subgroup consisting of half the elements of S_5. Namely, the group of sense-preserving symmetries of the dodecahedron is isomorphic to A_5.

Observation 2.10. The dodecahedron is a double cover of Petersen's graph, just like the sphere is a double cover of the projective plane.

Observation 2.11. The automorphism of the dodecahedron that sends every vertex to the opposite vertex (corresponding to multiplication by the matrix $-I$) commutes with all other symmetries of the dodecahedron (and also induces the trivial symmetry of Petersen's graph under the double cover mentioned above). However, the group S_5 has no element that commutes with every other group element. In particular, the automorphism group of the dodecahedron cannot be isomorphic to S_5.

Observation 2.12. A 3-dimensional solid may have two kinds of symmetries (congruences): the orientation-preserving symmetries and the orientation-reversing symmetries. Since the composition of any two symmetries of the same kind is an orientation-preserving symmetry, the set of orientation-preserving symmetries forms a subgroup of the symmetry group of the solid.

Problem 158. If X is a regular graph of degree r and girth at least 5, then X has at least $r^2 + 1$ vertices.

Let us find regular graphs of degree r with girth at least 5, and $r^2 + 1$ vertices, for small values of r.

(1) Let $r = 1$. Then we have the graph of two vertices with an edge between them.
(2) Let $r = 2$. Then we have a pentagon.
(3) Let $r = 3$. Then we have Petersen's graph.
(4) Let $r = 4, 5, 6$. Such graphs do not exist.
(5) Let $r = 7$. There is a graph of this form; it is called the Hoffmann-Singleton graph. It has 50 vertices and contains a lot of copies of the Petersen graph.

In fact, the following theorem holds.

Theorem 2.13 (Hoffmann-Singleton theorem). *If X is a regular graph of degree r with girth ≥ 5 and $n = r^2 + 1$ vertices, then the only possible values of r are $1, 2, 3, 7, 57$.*

It is unknown whether such a graph exists when $r = 57$.

3. COMPLEX EIGENVALUES OF REAL MATRICES. REAL EIGENVALUES OF COMPLEX MATRICES.

Definition 3.1. A matrix is called *symmetric* if $A = A^t$.

Theorem 3.2. *If A is a symmetric real matrix, then all its (complex) eigenvalues are real.*

We will now compute the eigenvalues of rotations. Let ρ_θ denote the rotation of the plane by an angle θ. The map ρ_θ has the matrix $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. To compute the eigenvalues of R_θ, we will compute the determinant of the matrix $tI - R_\theta$, which is the characteristic polynomial f_{R_θ} of R_θ. Then f_{R_θ} is the following:

$$f_{R_\theta} = (t - \cos\theta)^2 + \sin^2\theta = t^2 - 2\cos\theta + 1.$$

Therefore $(\lambda - \cos\theta)^2 = -\sin^2\theta$, which means that $\lambda - \cos\theta = \pm i\sin\theta$. Therefore $\lambda_{1,2} = \cos\theta \pm i\sin\theta$.

Problem 159. Find an eigenbasis of R_θ over $\mathbb{C}$, and observe that it is independent of θ.

Definition 3.3. The *conjugate-transpose* of a complex matrix $A = (a_{ij})$ is the matrix $A^* = (\overline{a}_{ji})$. That is, we take the transpose of the matrix and then conjugate every element. The matrix A^* is also called the (Hermitian) adjoint of the matrix A.

Definition 3.4. A complex matrix A is called *self-adjoint* or *Hermitian* if $A = A^*$.

Problem 160. If A is a self-adjoint complex matrix, then show that all the (complex) eigenvalues of A are real.

Exercise 3.5. If A, B are complex matrices that can be multiplied, then $(AB)^* = B^* A^*$.

If $z = u + iv$ is a complex number, then
$$\overline{z} \cdot z = z \cdot \overline{z} = (u + iv)(u - iv) = u^2 + v^2 = |z|^2.$$
If a is a vector in $\mathbb{C}^n$, then $a^* \cdot a = \sum_{i=1}^{n} \overline{a}_i a_i$.

Definition 3.6. If $a \in \mathbb{C}^n$, then the *norm* of a is defined as $\sqrt{a^* a}$, and is denoted by $\|a\|$.

Definition 3.7. Vectors $e_1, \ldots, e_n \in \mathbb{C}^n$ are said to be *orthonormal* if $e_i^* e_j = \delta_{ij}$.

Observation 3.8. Orthogonality in $\mathbb{C}^n$ is different from orthogonality in $\mathbb{R}^n$.

Theorem 3.9 (Spectral theorem). *Every complex Hermitian matrix has an orthonormal eigenbasis. The same is true for real symmetric matrices.*