REU APPRENTICE CLASS #16

INSTRUCTOR: LÁSZLÓ BABAI
SCRIBE: BOWEI ZHENG

Tuesday, July 19, 2011

1. Hermitian dot product, unitary matrices

Problem 160. If A is a self-adjoint complex matrix, i. e., $A = A^*$, then all the complex eigenvalues of A are real numbers.

Definition 1.1. The *standard Hermitian dot product* in $\mathbb{C}^n$ is $a^*b = \sum_{i=1}^n \bar{a}_i b_i$.

Definition 1.2. For $a, b \in \mathbb{C}^n$ we say that $a \perp b$ if $a^*b = 0$.

Observation 1.3. $(\forall a \in \mathbb{C}^n)(a \perp a \text{ iff } a = 0)$.

Note: "iff" is shorthand for "if and only if."

Definition 1.4. An *orthonormal system* in $\mathbb{C}^n$ is $e_1, e_2, \ldots, e_k$ such that $e_i^* e_j = \delta_{ij}$, i. e., $(\forall i)(\|e_i\|^2 = 1)$ and $(\forall i \neq j)(e_i \perp e_j)$.

Exercise 1.5. Every orthonormal system is linearly independent.

We often refer to the coordinates with respect to an orthonormal basis (ONB) $e_1, \ldots, e_n$ as *Fourier coefficients*, and the expression of a vector v as $v = \sum_{i=1}^n \alpha_i e_i$ as the *Fourier expansion.*

Observation 1.6. The *Fourier coefficients* can be calculated using standard Hermitian dot product: $\alpha_j = e_j^* v$.

Observation 1.7. $\mathbb{C}^n$ has an orthonormal basis, namely, the standard basis $(1, 0, \cdots, 0)^*, (0, 1, 0, \cdots, 0)^*, \cdots, (0, \cdots, 0, 1$

Definition 1.8. A *unitary matrix* is a matrix of which the columns form an orthonormal basis of $\mathbb{C}^n$.

Notation 1.9. $U(n)$ denotes the set of unitary matrices, which is a subset of $M_n(\mathbb{C})$.

Observation 1.10. Let $A = [a_1, \cdots, a_n]$ where $a_i \in \mathbb{C}^n, i = 1, \cdots, n$. $A \in U(n)$ iff $(\forall i, j)(a_i^* a_j = \delta_{ij})$ iff $A^*A = I$.

Exercise 1.11. $A \in U(n)$ iff the rows of A form an orthonormal basis of $\mathbb{C}^n$.

Observation 1.12. Unitary matrices preserve dot product. In other words, if $A \in U(n)$ then $(\forall u, v \in \mathbb{C}^n)(u^*v = (Au)^*(Av))$.

Definition 1.13. A is *unitarily similar* to B $(A \sim_u B)$ if $\exists S \in U(n)$ such that $B = S^{-1}AS = S^*AS$.

Theorem 1.14. *$(\forall A \in \mathbb{C}_{n\times n})(\exists$ upper triangular $T \in M_n(\mathbb{C}))(A \sim_u T)$.*

Problem 161. Prove Theorem 1.14.

Theorem 1.15. *Every orthonormal system can be extended to an orthonormal basis.*

Exercise 1.16. Pick an eigenvalue λ of $A \in M_n(\mathbb{C})$ and let b_1 be its normalized eigenvector. Extend b_1 to an orthonormal basis $b_1, \cdots, b_n$ by theorem 1.15. Let $[b_1, \cdots, b_n] := S \in U(n)$ and $S^{-1}AS = A' = [a'_1, \cdots, a'_n]$. Prove that $a'_1 = (\lambda, 0, \cdots, 0)^t$.

Exercise 1.17 (Multiplication of upper-triangular block matrices)**.** Note: the diagonal blocks must be square matrices, and the three matrices are identically partitioned.

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \\ 0 & 0 & B_{33} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ 0 & C_{22} & C_{23} \\ 0 & 0 & C_{33} \end{pmatrix}$$

where $(\forall i)(C_{ii} = A_{ii}B_{ii})$.

2. Normal Matrices, Orthogonal Matrices, Complex and Real Spectral Theorems

Definition 2.1. $A \in M_n(\mathbb{C})$ is *normal* if $A^*A = AA^*$.

Observation 2.2. Hermitian matrices ($A^* = A$), unitary matrices ($A^* = A^{-1}$), and diagonal matrices are all normal matrices.

Theorem 2.3 (Spectral Theorem')**.** *Let $A \in M_n(\mathbb{C})$. A is unitarily similar to a diagonal matrix (A is unitarily diagonizable) iff A is normal.*

Exercise 2.4. If $A \sim_u B$ and A is normal then B is also normal.

Problem 162. If a triangular matrix is normal, prove it is diagonal.

Problem 163. If A is unitary and λ is an eigenvalue of A, prove that $|\lambda| = 1$.

Problem 164. If A is normal, prove
1. A is Hermitian iff all eigenvalues of A are real.
2. A is unitary iff all eigenvalues of A have unit absolute value.

Observation 2.5. Theorem 2.3 is equivalent to saying that A is normal iff A has an orthonormal eigenbasis.

Theorem 2.6 (Real Spectral Theorem)**.** *If $A \in M_n(\mathbb{R})$ and A is symmetric, i. e., $A = A^t$, then A has an orthonormal eigenbasis (over $\mathbb{R}$).*

Definition 2.7. $B \in M_n(\mathbb{R})$ is an *orthogonal matrix* if $B^t = B^{-1}$, i. e., if the columns of B forms an orthonormal basis. In other words, B is a real unitary matrix. $O(n)$ denotes the set of orthogonal matrices; so $O(n) = M_n(\mathbb{R}) \cap U(n)$.

Definition 2.8. Let $A, B \in M_n(\mathbb{R})$. A is orthogonally similar to B ($A \sim_o B$) if there exists an orthogonal matrix $S \in O(n)$ such that $B = S^tAS$.

Problem 165. Let $A \in M_n(\mathbb{R})$. Prove that A is similar to a triangular matrix iff A is orthogonally similar to a triangular matrix iff all (complex) eigenvalues of A are real.

Problem 166 (Real Spectral Theorem)**.** Let $A \in M_n(\mathbb{R})$. Prove that A is orthogonally similar to diagonal matrix iff A is symmetric ($A = A^t$).

Problem 167. Prove A is an orthogonal matrix iff A is orthogonally similar to a block-diagonal matrix of the following form: each diagonal block is 1×1 or 2×2; the 1×1 blocks are ± 1; and the 2×2 blcoks are rotation matrices of the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \forall i = 1, \cdots, n.$

Definition 2.9. Let A be a real symmetric matrix, $x \in \mathbb{R}^n, x \neq 0$. Then the *Rayleigh quotient* of A at x is $\frac{x^tAx}{x^tx} := R(x)$.

Observation 2.10. If $x \neq 0 \in \mathbb{R}^n$ then $x^tx \neq 0 \in \mathbb{R}$. So the Rayleigh quotient is well defined.

Problem 168. Let the eigenvalues of the real symmetric matrix A be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Prove $\lambda_1 = \max_{x\in\mathbb{R}} R(x)$ and $\lambda_n = \min_{x\in\mathbb{R}} R(x)$.

3. Real Euclidean Space, Gram-Schmidt Orthogonalization, Complex Hermitian Space

Definition 3.1. A vector space V over $\mathbb{R}$ with an inner product $V \times V \to \mathbb{R}$, $a, b \mapsto \langle a, b\rangle$ is a *real Euclidean space* if the inner product satisfies the following conditions
1. (bilinear)
   - (a) $\langle a_1 + a_2, b\rangle = \langle a_1, b\rangle + \langle a_2, b\rangle$.
   - (b) $\langle \lambda a, b\rangle = \lambda\langle a, b\rangle$.
   - (c) $\langle a, b_1 + b_2\rangle = \langle a, b_1\rangle + \langle a, b_2\rangle$.
   - (d) $\langle a, \lambda b\rangle = \lambda\langle a, b\rangle$.
2. (symmetric) $\langle a, b\rangle = \langle b, a\rangle$
3. (positive definite) $\langle a, a\rangle > 0$ unless $a = 0$

Definition 3.2. The *standard inner product* on $\mathbb{R}^n$ is defined as $\langle a, b\rangle = a^t b$.

Definition 3.3. A real matrix B is *positive definite* if $B = B^t$ and $(\forall x \neq 0)(x^t Bx > 0)$.

Observation 3.4. An inner product on $\mathbb{R}^n$ can be obtained in the form $\langle a, b\rangle = a^t Bb$ where B is a positive definite matrix.

Problem 169. $B = B^t$ is a positive definite matrix iff all eigenvalues of B are positive.

Problem 170. $B = B^t$ is a positive definite matrix iff all the corner determinants of B are positive, i. e., $(\forall k \leq n)(\det((b_{ij})_{i,j\leq k}) > 0$.

Observation 3.5. For the set of continuous function on $[0, 1]$, denoted by $C[0, 1]$, we can define an inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$.

Observation 3.6. For the set of continuous functions on any finite or infinite interval I, a positive weight function $\mu(x), I \to \mathbb{R}^+$, we can define an inner product $\langle f, g\rangle = \int_I f(x)g(x)\mu(x)dx$ assuming $(\forall n)(\int_I x^{2n}\mu(x)dx) < \infty$.

Problem 171. Prove that $1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots \in C[0, 2\pi]$ is an orthogonal system under the inner product $\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)dx$. Infer that these trig. functions are linearly independent.

Definition 3.7. The *norm* of a vector a in a real Euclidean space is $\sqrt{\langle a, a\rangle} := \|a\|$.

Definition 3.8. We say that a, b are orthogonal ($a \perp b$) if $\langle a, b\rangle = 0$.

Definition 3.9. Let V be a real Euclidean space. The (real) *Gram-Schmidt Orthogonalization* is a process that converts the input $v_1, v_2, \cdots \in V$ to the output $b_1, b_2, \cdots \in Vn$ such that

(1) $(\forall i \neq j)(b_i \perp b_j)$.
(2) $(\forall k)(\text{span}(v_1, \cdots, v_k) = \text{span}(b_1, \cdots, b_k)$. Let us denote this subspace by V_k.
(3) $(\forall k)(b_k - v_k \in V_{k-1})$.

Theorem 3.10. $\forall v_1, v_2, \cdots$, *there exist unique* $b_1, b_2, \cdots$ *satisfying the conditions of the Gram-Schmidt Orthogonalization.*

Corollary 3.11. *Any orthogonal system that does not contain* 0 *can be extended to orthogonal basis.*

Observation 3.12. $b_k = 0$ iff $V_k = V_{k-1}$ iff $v_k \in \text{span}(v_1, \ldots, v_{k-1})$.

Problem 172. During the Gram-Schmidt orthogonalization process, we have $v_k - b_k = \sum_{j=1}^{k-1} \alpha_{kj} b_j$. Prove that

$$\alpha_{ki} = \frac{\langle b_i, v_k\rangle}{\|b_i\|^2}, \forall k = 1, 2, \cdots, i = 1, \cdots, k-1$$

.

Definition 3.13. A *complex Hermitian space* is a vector space over $\mathbb{C}$ with inner product satisfying all the properties of real Euclidean space with the modification

$1(b)^*$ $\langle \lambda a, b\rangle = \bar{\lambda}\langle a, b\rangle$.
2^* $\langle b, a\rangle = \overline{\langle a, b\rangle}$.

Problem 173. State and prove the Gram-Schmidt orthogonalization theorem in complex Hermitian space case.

Observation 3.14. In a real Euclidean space with orthonormal basis $b_1, \cdots, b_n$, $v = \sum_{i=1}^n \alpha_i b_i$, $w = \sum_{i=1}^n \beta_i b_i$, then $\alpha_i = \langle b_i, v\rangle$ and $\beta_i = \langle b_i, w\rangle$. Moreover, $\langle v, w\rangle = \sum_{i=1}^n \alpha_i\beta_i = [\alpha]^t[\beta]$.

Corollary 3.15. *If* V *is a Euclidean space,* $\dim V = n$*, then* V *is isometric to* $\mathbb{R}^n$ *with standard dot product.*