REU APPRENTICE CLASS #18

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1. Adjacency matrix, eigenvalues of undirected graphs

Problem 178. Let $A, B \in M_n(\mathbb{C})$. Assume $AB = BA$. Prove that they have a common eigenvector.

Problem 179. Let $A, B \in M_n(\mathbb{R})$, $A = A^t, B = B^t$ and $AB = BA$. Prove that they have a common orthonormal eigenbasis.

Exercise 1.1. Suppose $f(x) = x^4 + ax^3 + bx^2 + cx + 15$ with integer coefficients, i. e., $a, b, c \in \mathbb{Z}$. Suppose $k \in \mathbb{Z}$ is a root of $f(x)$, i. e., $f(k) = 0$. What values could k be? Narrow down the possibilities to a finite number of cases, independent of a, b, c.

Problem 180. Suppose $f(x), g(x) \in \mathbb{Z}[x]$ and $g(x)$ has leading coefficient 1. Prove the division $f(x) = g(x)q(x) + r(x)$ has integer coefficients quotient and remainder, i. e., $q(x), r(x) \in \mathbb{Z}[x]$.

Recall: If $A \in M_n(\mathbb{C})$ and λ_i are its eigenvalues then we have $\sum_i \lambda_i = \mathrm{Tr} A$ and $\prod_i \lambda_i = \det A$.

Definition 1.2. The *adjacency matrix* $A = (a_{ij})_{n\times n}$ of an undirected graph G with vertex set $\{1, \dots, n\}$ is the $n \times n$ matrix with $a_{i,j} = 1$ if i and j are adjacent and 0 otherwise.

Observation 1.3. The complete graph on n vertices, denoted by K_n, has $\binom{n}{2}$ edges.

Observation 1.4. Given n vertices, there are $2^{\binom{n}{2}}$ different possible graphs on these n vertices.

Observation 1.5. An undirected graph on n vertices has symmetric adjacency matrix and thus diagonalizable with n real eigenvalues, denoted by $\lambda_1 \geq \cdots \geq \lambda_n$.

Problem 181. Prove that $\frac{1}{n}\sum_{i=1}^n d(i) \leq \lambda_1 \leq \max_i d(i)$, where $d(i)$ denotes the degree of vertex i.

Notation 1.6. A_G denotes the adjacency matrix of the graph G and $f_G := f_{A_G}$ denotes the characteristic polynomial of the adjacency matrix A_G.

Exercise 1.7. If G, H are isomorphic graphs, then A_G is similar to A_H. In particular, $f_G = f_H$.

Observation 1.8.

$$A_{K_n} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} = J_n - I_n,$$

where

$$J_n = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

Observation 1.9. Suppose an $n \times n$ matrix A has eigenvalues $\lambda_1, \dots, \lambda_n$ (listed with multiplicity), then the matrix $A - I$ has eigenvalues $\lambda_1 - 1, \cdots, \lambda_n - 1$ with the same multiplicity for each eigenvalue of to A.

Observation 1.10. J_n has $n-1$ dimensional null space and thus has eigenvalue 0 with (geometric) multiplicity $n-1$. The remaining eigenvalue is n, using the trace of J_n. Hence, $f_{J_n}(t) = t^{n-1}(t-n)$ and thus $f_{K_n}(t) = (t+1)^{n-1}(t+1-n)$.

Problem 182. Suppose A has eigenvalues $\lambda_1, \cdots, \lambda_n$. Prove that $aA + bI$ has eigenvalues $a\lambda_i + b$ with corresponding multiplicities.

Observation 1.11. Suppose $B = \begin{pmatrix} a & b & \cdots & b \\ b & a & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \cdots & b & a \end{pmatrix} = bJ_n + (a-b)I_n$, then B has eigenvalue $a - b$ with multiplicity $n-1$ and another eigenvalue $(n-1)b + a$ with multiplicity 1. Hence, $f_B(t) = (t - (a - b))^{n-1}(t - (n-1)b - a)$.

Observation 1.12. If G is a regular graph of degree r (every vertex has degree r), then r is an eigenvalue with eigenvector $(1, \ldots, 1)^t$.

Problem 183. Assume A is a nonnegative matrix with a positive eigenvector x (all coordinates of x are positive) with eigenvalue λ, i. e., $x \neq 0$ and $Ax = \lambda x$. Prove $(\forall$ eigenvalue μ $)(|\mu| \leq \lambda)$.

Exercise 1.13. If a nonnegative symmetric matrix has a positive eigenvector, then all eigenvectors corresponding to other eigenvalues have some negative coordinates.

Exercise 1.14. If x is a nonnegative eigenvector of the connected graph G then x is strictly positive.

Problem 184. Suppose an undirected graph has sorted eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Prove

(1) $(\forall i)(|\lambda_i| \leq \lambda_1)$
(2) If the graph G is connected, then $(\forall i \geq 2)(\lambda_i < \lambda_1)$
(3) If the graph G is connected, then $|\lambda_n| = \lambda_1$ iff G is bipartite.
(4) If G is a bipartite graph, then $(\forall i)(\lambda_i = -\lambda_{n-i+1})$.

Problem 185. Let $g \in \mathbb{C}[x]$ and $A \in M_n(\mathbb{C})$. Assume A has eigenvalues $\lambda_1, \cdots, \lambda_n$ (listed with multiplicity, i. e., $f_A(t) = \Pi_{i=1}^n (t - \lambda_i)$). Prove that the eigenvalues of $g(A)$ are $g(\lambda_1), \cdots, g(\lambda_n)$ (again, listed with multiplicity).

Recall that we proved before, if a regular G with degree r has girth at least 5, then $n \geq r^2 + 1$. For such graph, if a and b are two vertices that are not connected, then they share a unique common neighbor. Next we have this amazing theorem.

Theorem 1.15 (Hoffman-Singleton)**.** *If a regular graph of degree $r \geq 1$ has girth at least 5 and $n = r^2 + 1$, then we can only have $r = \{1, 2, 3, 7, 57\}$*

Observation 1.16. K_2 represents $r = 1$. C_5 is the example for $r = 2$. Petersen's graph demonstrates the case $r = 3$. The "Hoffman – Singleton graph" shows $r = 7$ is possible. No example has been found for the case $r = 51$. It remains open.