REU APPRENTICE CLASS #19

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1. Gram matrix, volume

Definition 1.1. Let $v_1, \dots, v_k$ be k vectors in a Euclidean space. The *Gram matrix* of these k vectors, denoted by $G(v_1, \dots, v_k)$, is the $k \times k$ matrix $(\langle v_i, v_j \rangle)_{k\times k}$. The *Gram determinant* is the determinant of the Gram matrix.

Exercise 1.2. $\det G(v_1, \dots, v_k) = 0$ iff $v_1, \dots, v_k$ are linearly dependent.

Problem 186. Prove that the Gram matrix of a list of vectors is always positive semidefinite. Moreover, prove that the Gram matrix is positive definite iff the list of vectors is linearly independent.

Problem 187. Prove that $\det G(v_1, \dots, v_k) = \mathrm{Vol}_k(v_1, \dots, v_k)^2$ where Vol_k denotes the k-dimensional volume.

Observation 1.3. If $(\forall i \neq j)(v_i \perp v_j)$, then $\mathrm{Vol}_k(v_1, \dots, v_k) = \prod_{i=1}^k \|v_i\|$ and $G(v_1, \dots, v_k) = \mathrm{diag}(\|v_1\|^2, \dots, \|v_k\|^2)$. This solves Problem 187 for the case when the vectors are orthogonal. To prove the general case, solve the following exercise.

Exercise 1.4. Prove: Gram-Schmidt orthogonalization does not change the Gram determinant.

2. Counting spanning trees of a graph

Definition 2.1. A *spanning tree* of a graph with n vertices is a subgraph which is a tree that contains all n vertices.

Exercise 2.2. Every connected graph has a spanning tree. Every spanning tree of a graph with n vertices has $n - 1$ edges.

Observation 2.3. Let N_i denote the number of different spanning tree of the complete graph K_i. We have $N_1 = 1, N_2 = 1, N_3 = 3, N_4 = 4!/2 + 4 = 16, N_5 = 125, \dots$.

Theorem 2.4 (Cayley)**.** *The number of spanning trees of the complete graph K_n is n^{n-2}.*

Definition 2.5. The *Laplacian* of a graph G, denoted by L_G, is defined as $D_G - A_G$ where D_G is the diagonal matrix $\mathrm{diag}(\deg(1), \dots, \deg(n))$ with the degrees of the nodes on the diagonal and A_G is the adjacency matrix of the graph G.

Exercise 2.6. $\det L_G = 0$.

Problem 188. (1) Prove that all cofactors of L_G are equal.
(2) * (Matrix-Tree Theorem, Kirchhoff 1848) Each cofactor of L_G equals the number of spanning trees of G.
(3) Infer Cayley's formula from the Matrix-tree Theorem.

A proof of the Matrix-Tree Theorem can be found in the June 30 lecture notes of the instructor's 2005 REU course on the Abelian Sandpile Model.

3. Finite Markov Chains, mixing rate, eigenvalue gap

Suppose we have n states. Let X_t denote the particle's location among $\{1,\dots,n\}$ at time t. Define $p_{ij} = P(X_{t+1} = j \mid X_t = i)$.

Exercise 3.1 (t-step transition probabilities)**.** Let $p_{ij}^{(t)}) = P(X_{\ell+t} = i \mid X_\ell = j)$. Then $T^t = (p_{ij}^{(t)})_{n\times n}$.

Definition 3.2. The *transition matrix* T is the $n \times n$ matrix $(p_{ij})_{n\times n}$.

Definition 3.3. The *distribution* of a particle's location at time line t, denoted by $q_t = (q_{t1},\dots,q_{tn})$ where $q_{ti} = P(X_t = i)$.

Observation 3.4. $q_{ti} \geq 0, \quad \sum_{i=1}^n q_{ti} = 1$.

Observation 3.5. For a transition matrix $T = (p_{ij})$, we have $p_{ij} \geq 0$ and $(\forall i)(\sum_{j=1}^n p_{ij} = 1)$ (all row sums are zero). Such matrix is called a *stochastic matrix.*

Exercise 3.6. Prove: if A, B are stochastic matrices then AB is also a stochastic matrix.

Observation 3.7 (Evolution of the Markov Chain)**.** $q_{t+1} = q_t T$ by the "theorem of complete probability" (a property of conditional probabilities). Hence $q_t = q_0 T^t$.

Definition 3.8. The distribution q is *stationary* if $q = qT$, i. e., q is a left eigenvector to eigenvalue 1.

Problem 189. Prove: for the simple random walk on a connected graph, the stationary probability of node i is proportional to its degree $\deg(i)$.

Exercise 3.9. The right eigenvalues of a matrix are exactly the same as left eigenvalues of the same matrix. (However, the eigenvectors may differ.)

Theorem 3.10 (Perron-Frobenius)**.** *Suppose $A \in M_n(\mathbb{R})$ is a positive matrix, i. e., $(\forall i,j)(a_{ij} > 0)$. Then there exists a positive eigenvector.*

Problem 190. (a) Prove the Perron-Frobenius Theorem.

(b) Use the Perron-Frobenius Theorem to prove that every Markov Chain has a stationary distribution.

Notation 3.11 (Transition digraph)**.** Let G_T denote the digraph of possible transitions. The vertices of the digraph correspond to the states of the Markoov Chain; and there is $i \to j$ an edge (arrow) $p_{ij} \neq 0$.

Definition 3.12. A Markov chain is called ergodic if G_T is strongly connected and *aperiodic*, i. e., the period of the digraph (gcd of the lengths of all closed walks) equals 1.

Problem 191 (Mixing of ergodic Markov Chains)**.** If a Markov chain is ergodic, then the limit $\lim_{t\to\infty} T^t = L$ exists.

Exercise 3.13. If the limit matrix L exists then every row of L is a stationary distribution. If the Markov Chain is ergodic then the stationary distribution is unique and therefore all rows of L are identical.

The convergence to L is called "mixing;" and the rate of convergence the "mixing rate" of the Markov Chain. The mixing rate is a major current subject of study.

Let $\mu_1 = 1, \mu_2, \dots \mu_n$ be the (in general, complex) eigenvalues of the transition matrix of a Markov Chain. Under fairly general circumstances, the rate of convergence to stationary distribution is controlled by the gap between $\mu_1 = 1$ and $\max\{|\mu_2|,\dots,|\mu_n|\}$, referred to as the *eigenvalue gap.* Theorem 3.17 below formalizes this general phenomenon for the case of the simple random walk on an r-regular graph. Note that in this case $\mu_i = \lambda_i / r$ where λ_i is the i-th eigenvalue of the adjacency matrix of the graph.

Observation 3.14. For a connected regular graph of degree r the transition matrix of the simple random walk is $T = \frac{1}{r}A$ where A is the adjacency matrix.

Notation 3.15. Let A be the adjacency matrix of a connected regular graph of degree r. Let A have eigenvalues $\lambda_1 = r \geq \dots \geq \lambda_n$. Let $\lambda := \max\{|\lambda_2|,\dots,|\lambda_n|\}$.

Exercise 3.16. We know that $r \geq \lambda$. Prove that $r = \lambda$ if and only if either G is disconnected (in this case, $\lambda_2 = r$, or G is bipartite (in this case, $\lambda_n = -\lambda_2$). These are precisely the cases when our random walk is not ergodic. In all other cases, $\lambda < r$.

Theorem 3.17 (Mixing of simple random walk on a regular graph)**.** *Let G be a regular graph of degree r with eigenvalues $\lambda_1 = r \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $\lambda = \max\{|\lambda_2|, \ldots, |\lambda_n|\}$. For the simple random walk on a regular graph of degree r we have $|p_{ij}^{(t)} - \frac{1}{n}| \leq \left(\frac{\lambda}{r}\right)^t$.*

Problem 192. Use the Spectral Theorem to prove Theorem 3.17.

(The proof was given in class.)

Definition 3.18 (Operator norm)**.** Suppose $A \in M_n(\mathbb{R})$. Then

$$\|A\| := \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

Observation 3.19. By definition, $\|Ax\| \leq \|A\|\|x\|$.

Problem 193 (Operator norm)**.** (a) If $A = A^t$, prove $\|A\| = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$.
(b) Prove $(\forall A)(\|A\| = \sqrt{\lambda_{\max}(A^T A)})$.

Exercise 3.20 (Cauchy-Schwarz)**.** Prove $|a^t \cdot b| \leq \|a\|\|b\|$ and $\|Ax\| \leq \|a\|\|x\|$.