REU APPRENTICE CLASS #4

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1. Matrices

Problem 36. Let

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with $|a| < 1$ and $|d| < 1$. Show that

$$\lim_{k\to\infty} A^k = 0.$$

Problem 37. Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

What is A^k?

Exercise: matrix multiplication is associative and distributive. Verify it; that is, check that

$$(AB)C = A(BC),$$

$$A(B + C) = AB + AC$$

and

$$(A + B)C = AC + BC.$$

Homogeneous systems of equations are those of the form

$$\underline{A} \cdot \underline{x} = \underline{0}.$$

The solution set is

$$U = \{\underline{x} \in F^n \mid \underline{A} \cdot \underline{x} = \underline{0}\} \subseteq F^n,$$

which is a subspace of F^n. We showed that

$$\dim U = n - \text{rank}A.$$

Definition 1. The *transpose* of a matrix $A = (\alpha_{ij})$ is $A^T = (\alpha_{ji})$.

Observe that $(AB)^T = B^T A^T$.
Useful fact: If A is $r \times s$ and $B = (\underline{b}_1, \dots, \underline{b}_t)$ is $s \times t$, then

$$A \times B = (A\underline{b}_1, \dots, A\underline{b}_t).$$

Verify this!

2. Directed Graphs

Definition 2. A *directed graph* (also called a *digraph*) is a set of "vertices"

$$V = \{v_1, \ldots, v_n\}$$

with a set of "egdes" (ordered pairs, arrows)

$$E = \{(v_a, v_b), \ldots, (v_t, v_u)\}.$$

between them.

Definition 3. Each directed graph can be represented by a matrix, called the *adjacency matrix.* If there are n vertices in the graph, the adjacency matrix is an $n \times n$ matrix. If we call this matrix $A = (a_{ij})$, then $a_{ij} = 1$ if there is an edge from i to j, and 0 otherwise.

Problem 38. Let A be the adjacency matrix of a directed graph. Show that the (ij)-entry of A^t counts the t-step walks from i to j. Note that a *walk* is allowed to repeat vertices and edges, unlike a *path*!

3. Finite Markov Chains

Definition 4. A *stochastic matrix* is an $n \times n$ matrix $T = (p_{ij})$ where $p_{ij} \geq 0$ (all entries are nonnegative), and

$$\forall i \left(\sum_j p_{ij} = 1 \right)$$

(that is, each row sums to 1).

A stochastic matrix defines the transition probabilities for a particle moving from state to state (vertex to vertex). Let X_t denote the location of the particle at time t. (This is always one of the vertices).

$$p_{ij} = P(X_{t+1} = j \mid X_t = i),$$

that is, p_{ij} gives the probability that the particle will be at vertex j at time t given that it was in location i at time t. We can define the ℓ-step transition probability as

$$p_{ij}^{(\ell)} = P(X_{t+\ell} = j \mid X_t = i).$$

Problem 39. Verify that

$$\left(p_{ij}^{(\ell)}\right) = T^{\ell};$$

that is, taking the ℓ-th power of the one-step transition matrix of a finite Markov Chain gives us the ℓ-step transition matrix.

4. Linear Maps and Matrices

Given a basis $\underline{b} = (b_1, \ldots, b_n)$, we can assign coordinates to each vector with respect to that basis. We write these coordinates as a column vector.

Definition 5. We write $[v]_{\underline{b}} \in F^n$ to mean the column vector of coordinates of v in the basis $\underline{b}$.

Definition 6. Let V and W be vector spaces over the same field F. Let

$$\phi : V \to W.$$

We say that ϕ is a *linear function* or *linear map* if it preserves linear combinations; that is,

$$(\forall x, y \in V)\, \phi(x + y) = \phi(x) + \phi(y)$$

and

$$(\forall v \in V)(\forall \alpha \in F)\, (\phi(\alpha x) = \alpha\phi(x)).$$

Theorem 1. *If $b_1, \dots, b_n$ is a basis for V and $w_1, \dots, w_n$ are any vectors in W, then there is a unique linear map*

$$\phi : V \to W$$

such that

$$(\forall i)(\phi(b_i) = w_i).$$

Definition 7. (The matrix associated with a linear map) Let $\phi : V \to W$ be a linear map, $\underline{e} = (e_1, \dots, e_n)$ a basis in V, and $\underline{f} = (f_1, \dots, f_n)$ a basis in W. Let $[\phi]_{\underline{e},\underline{f}}$ denote the $k \times n$ matrix of which the j-th column is $[\phi(e_j)]_{\underline{f}}$, for $1 \le j \le n$.

Exercise: Show that

$$[\phi]_{\underline{e},\underline{f}} \cdot [v]_{\underline{e}} = [\phi(v)]_{\underline{f}}.$$

Also, if $\phi : x \mapsto Ax$, show that $[\phi]_{st_n, st_k} = A$ where st_n denote the standard basis of F^n.

Definition 8. A *linear transformation* is a linear map $\phi : V \to V$. In this case we write $[\phi]_{\underline{e}}$ instead of $[\phi]_{\underline{e},\underline{e}}$.

Let $\underline{e} = (e_1, e_2)$ be the standard basis in the plane (two perpendicular unit vectors). Let ρ_θ denote the rotation of the plane by angle θ about the origin (counterclockwise). Then

$$[\rho_\theta]_{\underline{e}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Let $\underline{f} = (f_1, f_2)$ be another basis in the plane, defined by taking $f_1 = e_1$ and $f_2 = \rho_\theta(f_1)$. Then

$$[\rho_\theta]_{\underline{e}} = \begin{pmatrix} 0 & -1 \\ 1 & 2\cos\theta \end{pmatrix}$$

We observed that these two matrices have the same determinant (1) and the same trace $(2\cos\theta)$.