REU APPRENTICE CLASS #5

INSTRUCTOR: LÁSZLÓ BABAI
SCRIBE: MATTHEW WRIGHT

Friday, July 1, 2011

1. Matrices and linear transformations

Exercises. Prove:

$$\text{rank}(A + B) \leq \text{rank}(A)$$

and

$$\text{rank}(AB) \leq \text{rank}(A);$$

we can then use the transpose to show that

$$\text{rank}(AB) \leq \text{rank}(B).$$

Theorem 1 (First Miracle of Linear Algebra)**.** *If $v_1, \ldots, v_k$ are linearly independent and for each i we have*

$$v_i \in \text{Span}(w_1, \ldots, w_\ell)$$

then $k \leq \ell$.

Recall: when we fix bases for V and W, each linear map

$$\phi : V \to W$$

can be associated (uniquely) with a matrix. Composing linear maps exactly corresponds to multiplication of their matrices, which shows that we basically had to define matrix multiplication in the way we did!

We can now use the rotation matrices to derive the basic trig identities:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

because a rotation by α composed with a rotation by β gives a rotation by $\alpha + \beta$. Expanding the matrix multiplication gives us the angle sum formulas

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

and

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Recall that the space $\mathbb{R}^{\leq n}[x]$ of polynomials of degree up to n has dimension $n + 1$. The derivative

$$\frac{d}{dx} : f \mapsto f'$$

is a linear transformation of this space, so we can write an $(n+1)\times(n+1)$ matrix for it (with respect to the standard basis) in the usual way of writing a matrix whose columns are the coordinates of the images of the basis vectors:

$$\left[\frac{d}{dx}\right]_{\{1,x,x^2,x^3,\dots,x^n\}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Definition 1. A linear map

$$\phi : V \to W$$

is an *ismorphism* if it is a bijection from V onto W.

Exercise: Show that if ϕ is an isomorphism, then so is the inverse of ϕ.

Theorem 2. *An isomorphism takes linearly independent sets to linearly independent sets.*

Exercise: Show that if ϕ is an isomorphism, then for any list of vectors $v_1, \dots, v_k \in V$ we have

$$\operatorname{rank}(v_1, \dots, v_k) = \operatorname{rank}(\phi(v_1), \dots, \phi(v_k)).$$

Definition 2. We say that vector spaces V and W are *isomorphic* if there is an isomorphism between them. In this case we write $V \cong W$.

Note that the isomorphism relation is an equivalence relation. That is, it satisfies the following properties:

(1) Reflexive: X is isomorphic to X;
(2) Symmetric: if X and Y are isomorphic, then Y and X are isomorphic;
(3) Transitive: If X and Y are isomorphic and Y and Z are isomorphic, then so are X and Z.

Theorem 3. *If V is a vector space over F and $\dim V = n$, then $V \cong F^n$.*

This implies that *any* two vector spaces of dimension n over the same field are isomorphic, and so a vector space (up to isomorphism) is specified by just two things: the field, and the dimension.

Do vector spaces of different dimension have to be non-isomorphic? Yes! If F^{13} and F^7 were isomorphic (for example), then because F^{13} has 13 linearly independent vectors F^7 would have to have that many as well, which it doesn't, by the First Miracle.

Definition 3. Let $\phi : V \to W$.

The *kernel* of ϕ is the subspace of V defined by

$$\begin{aligned} \ker\phi &= \phi^{-1}(0_W) \\ &= \{v \in V \mid \phi(v) = 0\}. \end{aligned}$$

The *image* of ϕ is the subspace of W defined by

$$\operatorname{Im}\phi = \{\phi(v) \mid v \in V\}.$$

Exercise: Verify that $\ker\phi$ is a subspace of V, and that $\mathrm{Im}(\phi)$ is a subspace of W.

For example, if π is the projection map from the plane to a line, then $\mathrm{Im}(\pi)$ is the line we're projecting on, and $\ker(\pi)$ is a line perpendicular to it. For rotations, we have $\ker(\rho_\theta) = \{0\}$ and $\mathrm{Im}(\rho_\theta) = \mathbb{R}^2$. If we now let π be a projection from $\mathbb{R}^3$ to the xy-plane, $\ker(\pi)$ is the z axis and $\mathrm{Im}(\pi)$ is the xy-plane.

Note that in each case the dimension of the kernel and the dimension of the image add up to the dimension of V. This is true in general:

Theorem 4. *If $\phi : V \to W$, then*

$$\dim\ker(\phi) + \dim\mathrm{Im}(\phi) = \dim V.$$

Note that the dimension of W is not involved.

Going back to the derivative map from $\mathbb{R}^{\leq n}[x]$ to itself,

$$\ker\left(\frac{d}{dx}\right) = \{\text{constants}\}.$$

$$\mathrm{Im}\left(\frac{d}{dx}\right) = \mathbb{R}^{\leq n-1}[x].$$

This gives an example of where the kernel is a subset of the image. We can also find an example of where the image is a subset of the kernel; sending everything to 0 gives one example, but here's a less trivial example on $\mathbb{R}^2$: send

$$\begin{aligned} e_1 &\mapsto 0 \\ e_2 &\mapsto e_1. \end{aligned}$$

The image and kernel of this map are both $\mathrm{Span}(\mathrm{e}_1)$.

Recall that we can write

$$\underline{A} \cdot \underline{x} = \underline{0}$$

as a shorthand for a homogeneous system of linear equations. Let

$$\phi : F^n \to F^k$$

denote the map $x \mapsto Ax$. Note that the *solution space* of the system $Ax = 0$ is exactly the kernel:

$$U = \{x \mid Ax = 0\} = \ker\phi.$$

So

$$\dim U = \dim\ker\phi = n - \dim\mathrm{Im}\,\phi.$$

The image of ϕ is exactly the column space of A:

$$\mathrm{Im}\,\phi = \{Ax \mid x \in F^n\} = \mathrm{col}(A),$$

whose dimension is the rank of A. This gives a formal proof that

$$\dim(U) = n - \mathrm{rank}A.$$

Definition 4. The *nullity* of A is the dimension of the kernel of ϕ, which is also the dimension of the solution space for A.

Theorem 5 (Rank/Nullity Theorem)**.** *For any $k \times n$ matrix A,*

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = n.$$

2. Determinants, permutations

The determinant is a function

$$\det : F^{n\times n} \to F.$$

In the case of 2×2 matrices,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

For 3×3 matrices it's more complicated:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \alpha_{11}\alpha_{22}\alpha_{33}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}\alpha_{23}\alpha_{32}-\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{13}\alpha_{22}\alpha_{31},$$

and for 4×4 we'll have 24 terms.

Definition 5. A *permutation* of a set S is a bijection from $S \to S$.

A permutation can be described by a table, specifying where each element goes. For example, here's a permutation $\pi : \{1, 2, 3\} \to \{1, 2, 3\}$:

x	1	2	3
$\pi(x)$	2	1	3

Definition 6. The *sign* of a permutation π is

$$\operatorname{sgn}(\pi) = (-1)^{\operatorname{Inv}(\pi)},$$

where

$$\operatorname{Inv}(\pi) = \text{number of inversions of } \pi,$$

that is, the number of pairs $a < b$ with $\pi(a) > \pi(b)$.

If $\operatorname{sgn}(\pi) = 1$ we say that π is even; otherwise we say it's odd.

Problem 40. Show that

$$\operatorname{sgn}(\pi\sigma) = \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma).$$

For example, consider the permutation

x	1	2	3	4	5
$\pi(x)$	3	5	1	4	2

This has 7 inversions, so it's an odd permutation. We can follow where each number goes:

$$1 \to 3 \to 1, \quad 4 \to 4, \quad 2 \to 5 \to 2.$$

Every permutation creates disjoint cycles like this.

We'll write $[n]$ as a shorthand for $\{1, 2, \ldots, n\}$.

Definition 7. The *symmetric group* S_n is the group of permutations of $[n]$. So $|S_n| = n!$.

A *transposition* is a 2-cycle, i. e., a permutation that switches two elementsand fixes everything else. All transpositions are odd permutations. We can build any permutation as a product of transpositions.

Exercise: Prove that every permutation can be expressed as a sequence of transpositions. That is, the transpositions generate S_n.

Corollary 1. *A permutation π is even if and only if it's the product of an even number of transpositions.*

Problem 41. Prove directly, without using the previous problem, that the product of an odd number of transpositions can never be the identity.

Some notation:

(1) We can write each cycle by specifying the elements in the order we follow them, so if we had a permutation with the cycle
$$1 \to 4 \to 3 \to 2 \to 1$$
we would write (1432) or (4321) or (3214) or (2143). These four expressions denote the same 4-cycle. Every permutation is the unique product of disjoint cycles; for instance, the permutation described before Definition 7 can be written in cycle notation as (13)(25) or equivalently as (31)(25) or (13)(52) or (31)(52). (We omit the 1-cycle (4).)

(2)

Using the parity of permutations, we can define the determinant:
$$\det A = \sum_{\pi \in \mathrm{perm}(1,2,\ldots,n)} \mathrm{sgn}(\pi) a_{1,\pi(1)}, a_{2,\pi(2)} \cdots a_{n,\pi(n)}.$$