REU APPRENTICE CLASS #7

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Problem session

Today we solved the problems 13, 22(b), 27, 28, 35, 36, 37, 40, 41(a), 42, and 44. Here are some facts worth remembering.

There is a nice geometric proof for the fact that the sum of the n-th complex roots of unity is zero for $n > 1$. Namely, this sum is invariant under rotation around the origin by the angle $\theta = 2\pi/n$. If $n > 1$, this rotation has the origin as its unique fixed point.

While solving Problem 28 we noticed how choosing a good notation (e.g., writing ω instead $e^{2\pi i/n}$) helps with the thinking and the exposition.

The solution to Problem 37 involved the Fibonacci numbers F_n, namely, for

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

the k-th power of A is given by

$$A^k = \begin{pmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{pmatrix}$$

From this, and from the fact that $A^{k+\ell} = A^k A^\ell$ we immediately infer the useful and nontrivial identity

$$F_{k+\ell} = F_k F_{\ell-1} + F_{k+1} F_\ell.$$

From the solution of Problem 22(b) we found the formula

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

for the n-th Fibonacci number. This shows in particular that the asymptotic equality

$$F_n \sim \frac{\varphi^n}{\sqrt{5}}$$

holds, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. In fact, F_n is equal to the nearest integer to $\frac{\varphi^n}{\sqrt{5}}$.

While discussing Problem 44, we also asked whether or not a union of subgroups is a subgroup. We showed that for $A, B \leq G$, $A \cup B$ is a subgroup if and only if $A \subseteq B$ or $B \subseteq A$.

To solve Problem 40 we used the formula

$$\operatorname{sgn}(\sigma) = \prod_{1 \leq i < j \leq n} \frac{\sigma(i) - \sigma(j)}{i - j}$$

for the sign of a permutation. To see that this coincides with the original definition, we just have to check that both expression have the same sign and the same absolute value.