Discrete Math, 22th day, Friday 8/13/04
REU 2004. Info: http://people.cs.uchicago.edu/∼laci/reu04.

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1 $0, 1$-measures

Exercise 22.1. Prove the following theorem. (*Hint:* Use Zorn's Lemma.)

Theorem 22.2. *If Ω is an infinite set, then there exists a finite additive nontrivial $0, 1$ measure on 2^Ω, $\mu : 2^\Omega \to \{0, 1\}$, such that $\mu(\Omega) = 1$. Nontrivial means that $\mu(\{a\}) = 0$ for all a.*

This is a hint to the infinite switch problem.

2 Sign-Rigidity (Paturi-Simon)

Let M be a *sign matrix,* i.e., a matrix consisting of ± 1 entries A matrix A of the same dimensions as M *realizes* M if the sign of $a_{i,j}$ is equal to $m_{i,j}$. The *sign-rank* of M is the minimum rank of a matrix realizing M.

Exercise 22.3 (Alon-Frankl-Rödl, 1984). Use Warren's theorem to prove that almost all $n \times n$ sign-matrices have sign-rank greater than or equal to $\frac{n}{32}$.

Finding explicit matrices that satify the preceding example is a hard problem. No particular examples are known.

Conjecture 22.4. *Hadamard matrices have sign-rank $\geq cn$.*

For applications, all we need would be an explicit matrix with rank greater than n^ϵ for some fixed $\epsilon > 0$. The best known explicit bound was $\Omega(\log n)$ until 2002.

Theorem 22.5 (Forster 2002). *Let $X \subseteq \mathbb{R}^k$ such that $|X| \geq k$ and the elements of X are in general position (i.e., k-wise linearly independent). Then there exists an invertible $k \times k$ matrix A such that $\sum_{x \in X} \frac{(Ax)(Ax)^T}{(Ax)^T(Ax)} = aI_k$, a constant multiple of the identity matrix.*

If we assume that the theorem is true, the value of a can easily be found by taking the trace on both sides (recall that the trace of the product of two matrices is independent of the order). This gives $ak = \sum_{x \in X} 1 = |X|$

Exercise 22.6 (Forster). If A realizes M, then $\text{rk}(A) \geq \frac{n}{||M||}$. Recall that $||M|| = \max_{||x||=1} ||Mx|| = \sqrt{\lambda_{\max}(M^T M)}$ by the spectral theorem.

Corollary 22.7. *If H is a Hadamard matrix, then* $\text{sign-rank}(H) \geq \sqrt{n}$.

Proof: If A realizes H, then $\text{rk}(A) \geq \frac{n}{||H||}$. Since $H^T H = nI$, we get that $||H||^2 = \lambda_{\max}(nI) = n$