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The gap between the chromatic number of a graph and the rank of its adjacency matrix is superlinear

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Abstract

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We present a sequence of graphs G_n for which $\chi(G_n) \geqslant \Omega(\mathrm{rk}(A(G_n))^{\frac{4}{3}})$.

1. Introduction

The question addressed in this paper is how efficiently the chromatic number $\chi(G)$ of a graph G might be estimated in terms of the rank of the adjacency matrix $A(G)$ of this graph. At one time it was thought that $\chi(G) \leqslant \mathrm{rk}(A(G))$ for all nontrivial graphs G. This conjecture was recently disproved by Alon and Seymour [1] who found out a sequence of graphs G_n for which $\chi(G_n) = \frac{32}{29}\mathrm{rk}(A(G_n))$. In this note we prove that the gap between $\chi(G)$ and $\mathrm{rk}(A(G))$ is superlinear by presenting graphs G_n on n^5 vertices with $\chi(G_n) \geqslant \Omega(n^4)$ and $\mathrm{rk}(A(G_n)) \leqslant \mathrm{O}(n^3)$.

The question under discussion is of especial interest in view of a connection with the communication complexity revealed by Lovász and Saks in [4]. Namely, they noted that the rank lower bound $\log_2 \mathrm{rk}(A)$ of Mehlhorn and Schmidt [5] for the (deterministic) communication complexity $\mathrm{DCC}(A)$ of a 0–1 matrix A is tight up to a polynomial *if and only if* $\chi(G) \leqslant \exp(\log(\mathrm{rk}(A(G)))^{\mathrm{O}(1)})$ for arbitrary graphs G. An immediate corollary of our result is an example of 0–1 matrices A_n such that $\mathrm{DCC}(A_n) \geqslant \frac{4}{3}\log_2 \mathrm{rk}(A_n) - \mathrm{O}(1)$. Actually even the stronger fact $\mathrm{NCC}(A_n) \geqslant \frac{4}{3}\log_2 \mathrm{rk}(A_n) - \mathrm{O}(1)$ holds where NCC stands for the *nondeterministic* communication complexity.

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2. The result

All graphs in this paper are undirected, without loops and multiple edges. $V(G)$ is the set of vertices of a graph G, $E(G)$ is the set of its edges. $\chi(G)$ is the chromatic number of G, $\alpha(G)$ is the size of the largest set of vertices which are mutually independent in G. K_n is the complete graph on n vertices. The *adjacency matrix* $A(G)$ of a graph G with $V(G)=\{v_1, \ldots, v_n\}$ is a 0–1 symmetric n to n matrix where $a_{ij}=1$ iff $(v_i, v_j) \in E(G)$. The *spectrum* $\mathbf{Sp}(G)$ of a graph G is the spectrum of $A(G)$ (over reals) considered as a multiset (i.e., all eigenvalues are taken with their multiplicities).

Define now a special sequence of graphs G_n. Let $V_1, \ldots, V_5$ be five disjoint sets, of cardinality n each. Set

$$V(G_n) \rightleftharpoons \textstyle\prod_{i=1}^5 V_i.$$

For $x,y \in V(G_n)$ $(x=(x_1, \ldots, x_5); y=(y_1, \ldots, y_5))$ define $\beta(x, y) \in \{0, 1\}^5$ as follows: $\beta(x, y)_i = 1$ iff $x_i \neq y_i$. We connect x and y by an edge of the graph G_n if and only if $\beta(x, y)$ belongs to the following set $\mathcal{B}$:

$$\mathcal{B} \rightleftharpoons \{0, 1\}^5 \setminus \{(00000), (11100), (11010), (11001), (11110), (11101), (11011), (00111)\}.$$

Theorem. (a) $\mathrm{rk}(A(G_n)) \leqslant O(n^3)$,
(b) $\chi(G_n) \geqslant \Omega(n^4)$.

Proof. (a) Note that G_n is the NEP-sum (see e.g. [2, Section 2.5]) of five copies of K_n with the basis $\mathcal{B}$. This allows us to evaluate $\mathbf{Sp}(G_n)$ in the form

$$\mathbf{Sp}(G_n) = \{f_{\mathcal{B}}(\lambda_1, \ldots, \lambda_5) \mid \lambda_i \in \mathbf{Sp}(K_n)\}, \tag{1}$$

where

$$f_{\mathcal{B}}(x_1, \ldots, x_5) \rightleftharpoons \sum_{\beta \in \mathcal{B}} \prod_{i=1}^5 x_i^{\beta_i}$$
$$= \prod_{i=1}^5 (1+x_i) - 1 - x_1x_2(x_3+x_4+x_5+x_3x_4+x_3x_5+x_4x_5) - x_3x_4x_5$$

(see e.g. [2, Theorem 2.23]). It is easy to check that

$$f_{\mathcal{B}}(-1, \ldots, -1) = 0 \quad \text{and} \quad \left.\frac{\partial f_{\mathcal{B}}}{\partial x_i}\right|_{(-1,\ldots,-1)} = 0 \quad \text{for } 1 \leqslant i \leqslant 5.$$

Since $f_{\mathcal{B}}$ is linear in each variable, it follows that $f_{\mathcal{B}}(x_1, \ldots, x_5)=0$ whenever at most one of $x_1, \ldots, x_5$ differs from (-1). But $\mathbf{Sp}(K_n) = \{(-1), \ldots, (-1), n-1\}$ ((-1) occurs $(n-1)$ times). Therefore, the number of points in $\mathbf{Sp}(K_n)^5$ which have at least two coordinates different from (-1) does not exceed $O(n^3)$. By (1) we have that $\mathbf{Sp}(G_n)$ contains at most $O(n^3)$ nonzero eigenvalues which exactly means $\text{rk}(A(G_n)) \leqslant O(n^3)$.

(b) It is sufficient to show that $\alpha(G_n) \leqslant O(n)$. For let S be an independent set of vertices in G_n. Given $I \subseteq \{1, \ldots, 5\}$, denote by p_I the natural projection $p_I : V \to \prod_{i \in I} V_i$. Let $S_I \rightleftharpoons p_I(S)$. Then it is easy to see that S_{12} is a matching in $V_1 \times V_2$ and hence $|S_{12}| \leqslant n$. If for each $\bar{x} \in S_{12}$ we have $|p_{12}^{-1}(\bar{x}) \cap S| \leqslant 3$ then the proof is completed. So, we may assume that there exists $\bar{x} \in S_{12}$ such that $|p_{12}^{-1}(\bar{x}) \cap S| \geqslant 4$. Let us see that in this case $S_{12} = \{\bar{x}\}$.

Indeed, consider $H \rightleftharpoons p_{345}(p_{12}^{-1}(\bar{x}) \cap S)$; $H \subseteq S_{345}$. Then H is a 3-matching of size $\geqslant 4$ in $V_3 \times V_4 \times V_5$. If $y \in S$ were a vertex for which $p_{12}(y) \neq \bar{x}$, then $p_{345}(y)$ should have a common vertex with each member of the 3-matching H (because otherwise y would be adjacent to the corresponding vertex in $p_{12}^{-1}(\bar{x}) \cap S$). That is impossible since the size of H is $\geqslant 4$.

So, we have $S_{12} = \{\bar{x}\}$ and hence $|S| = |H| \leqslant n$ because H is a 3-matching. □

The notion of the (deterministic) communication complexity DCC(A) of a 0–1 matrix A was introduced by Yao in his seminal paper [6]. Two efficient lower bounds for DCC(A) are known: the nondeterministic communication complexity NCC(A) [3] which equals $\lceil \log_2 \rceil$ of the smallest number of 1-rectangles one needs to cover all 1-entries in A and the rank lower bound $\log_2 \text{rk}(A)$ invented by Mehlhorn and Schmidt [5]. Lovász and Saks [4] asked whether the rank lower bound is optimal up to a polynomial. We can derive from the theorem above the following modest separation between NCC and $\log_2 \text{rk}$ (and hence also between DCC and $\log_2 \text{rk}$).

Corollary. *There are* 0–1 *matrices* A_n *for which*

$$\text{NCC}(A_n) \geqslant \tfrac{4}{3} \log_2 \text{rk}(A_n) - O(1).$$

Proof. Take $A_n \rightleftharpoons J - A(G_n)$ where J is the matrix with all entries equal 1 and G_n are the graphs from the theorem. Then $\text{rk}(A_n) \leqslant O(n^3)$ whereas $\text{NCC}(A_n) \geqslant 4 \log_2 n - O(1)$ because even to cover the diagonal of A_n one needs $\chi(G_n) \geqslant \Omega(n^4)$ 1-rectangles. □

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