It has become a good tradition among people working in the theory of Feasible Proofs and adjacent areas to run from time to time major workshops on the subject. They are well attended by the strongest researchers and serve as an excellent pretext to publish collections of papers afterwards. In fact, contributions to these collections are very generous and are to a large extent made by strong people that were not able to attend the event itself. As a consequence, the resulting volumes can be called "proceedings" only in a rather conventional way: they rather accumulate the state of art in the area of Feasible Proofs, stimulate initiative and suggest directions for further research. The first two books in this informal series were *Arithmetic, Proof Theory and Computational Complexity* (1993) and *Feasible Mathematics II. Progress in Computer Science and Applied Logic, vol.* 13 (1995); the book under review is the third enterprise.

It is necessary to precede the survey of individual papers from the collection with a brief introduction to the core concepts of the theory of Feasible Proofs itself. Fix a popular complexity class C like the class of poly-time computable functions $\mathbf{P}$ or the class $\mathbf{NC}^1$ of functions computable by poly-size "uniform" propositional formulae. Then the complexity theory studies which objects (languages, functions etc.) *exist* in the world of C-computability. Proof Complexity asks what can be *proved within this C-world* if we also require all *intermediate* concepts involved in the proof to be C-computed.

The most straightforward way of formalizing this idea is to find a "reasonable" equational theory E_C whose terms are exactly C-computable functions. The proof-theoretic view requires to present a fragment T_C of PA such that C coincides with the class of Σ_1^b-definable in T_C functions (Σ_i^b is some "bounded" analogue of the ordinary hierarchy Σ_i; for this reason the theories T_C are also known under the generic name of *Bounded Arithmetic*). The proof complexity in a theoretical computer scientist's sense perfectly fits the framework of *propositional proof systems*, i.e., sound and complete systems for the propositional calculus. We associate to C a propositional proof system P_C with the major requirement that admissible lines in a P_C-proof must essentially constitute formulae (or circuits) of the form specified by the class C. Thus, a P_C-proof "exists" in the C-world iff it has small (usually, polynomial) complexity (typically measured by the number of bits).

The success of the theory of feasible proofs is largely determined by the empirical circumstance that these three approaches turn out to be very similar, almost equivalent, and they are also heavily linked to purely computational issues about the C-world. This really allows us to talk of a *robust theory of C-provability.* And the typical robustness tests for the members of the triad $\langle E_C, T_C, P_C\rangle$ are these:

- E_C and T_C have a common extension conservative over both of them.
- There exists a translation of Σ_0^b-formulae $A(a)$ into a sequence of propositional formulae $A_n(p_1, \ldots, p_n)$ (expressing the validity of A on strings a of length n) such that $T_C \vdash A(a)$ implies that $A_n(p_1, \ldots, p_n)$ have efficient P_C-proofs. A similar translation holds for E_C vs. P_C.
- T_C and E_C prove the soundness of P_C. This implies that if $A_n(p_1, \ldots, p_n)$ have "sufficiently uniform" efficient P_C-proofs, then $T_C \vdash A(a)$, and similarly for E_C.

Comparing the content of the volume under review with two previous volumes mentioned at the beginning, one can observe a substantial drift to the framework of propositional proof complexity. This adequately reflects the changes happening in the community, and the reason for this tendency is mostly practical: of the three members in the triad $\langle E_C, T_C, P_C\rangle$, propositional proof systems provide by far more convenient and elegant framework for contemplating over independence results in the C-world. Eight (out of sixteen) papers in this volume are devoted to "pure" propositional proofs, and I begin surveying individual papers with them.

Resolution is the classical and one of the most widely studied proof systems. JUKNA proves an elegant lower bound for some general hitting principle in the (stronger) model of *semantic* resolution. This bound unifies several previously known major results and sheds light on the mechanisms of their proofs. FU considers *Cutting Planes* (this proof system extends Resolution by allowing to operate with arbitrary linear inequalities over integers). In this model, he proves a strong lower bound for one important combinatorial principle, *modulo r colouring principle.* Certain combinatorial statements related to the *pigeon-hole principle*, as well as some natural graph theoretic principles are studied by CLOTE and SETZER in the models of Resolution, Cutting Planes and some of their extensions.

The weakest propositional proof system for which no non-trivial lower bounds are currently known is the system $F_d(MOD_p)$ in the $\mathbf{ACC}^0[p]$-world (for a fixed prime p, $\mathbf{ACC}^0[p]$ consists of poly-size Boolean formulae that have constant logical depth and, along with ordinary connectives, allow also the modular connectives MOD_p of arbitrary arity). This is a very intriguing target since *computational* lower bounds for $\mathbf{ACC}^0[p]$ were proven over ten years ago. Respectively, the problem of showing matching lower bounds for $F_d(MOD_p)$ has become by now one of the central open questions in the area. In pursuing this goal, an elegant class of proof systems of a slightly different nature was introduced. These are algebraic proof systems based upon Hilbert's Nullstellensatz: in these proof systems we prove facts of the form "every solution to some system of polynomial equations $f_1 = \ldots = f_m = 0$ is also a solution to another equation $g = 0$". A Nullstellensatz proof simply consists of polynomials $g_1, \ldots, g_m$ such that $f_1g_1 + \cdots + f_mg_m = 1$, and its *complexity* is traditionally measured by the maximal degree of $g_1, \ldots, g_m$. Four papers in the volume are devoted to this line of research which justly reflects its importance.

BUSS gives an excellent survey of known results for the Nullstellensatz proof system. This survey specifically focuses on the concept of a *design* that has proved to be the major tool in obtaining lower bounds on the degree of Nullstellensatz proofs. In their very strong paper BEAME and RIIS construct a combinatorial design that allows them to prove the first lower bounds in the Nullstellensatz system for the *onto* version of the pigeon-hole principle. As one application of this result, they prove exponential lower bounds for this principle in the fragment of $F_d(MOD_p)$ that extends bounded depth Frege system F_d by modulo p counting principle. PUDLAK and SGALL establish the monotone version of effective interpolation for the Nullstellensatz system and its natural extension, *Polynomial Calculus.* Their result relates algebraic proof systems with the well-known computational model of span programs and certain extensions of the latter. MACIEL and PITASSI propose an approach to lower bounds for $F_d(MOD_p)$ that is in a sense alternative to algebraic proof systems. They translate one well-known in Complexity Theory simulation to the level of propositional proofs (which is highly non-trivial!) and show how to convert efficiently $F_d(MOD_p)$-proofs for arbitrary fixed depth d into *depth-3* proofs with threshold connectives of a special form.

There is little hope to prove with current methods absolute lower bounds for strong propositional proof systems like *Frege* or *Extended Frege* (corresponding to $\mathbf{NC}^1$ and $\mathbf{P}$, respectively). The best we can hope for at the moment is to analyze the *relative* strength of various combinatorial principles by showing their completeness under suitable reducibilities. The main obstacle on this way is the shortage of good examples which is in a striking contrast with the rich theory for computational classes. AVIGAD makes an extremely important (perhaps, even the first) step in the direction of filling this gap. He defines *combinatorial* tautologies that are *complete for EF over F* in the sense that any Frege system augmented with these tautologies becomes polynomially equivalent to Extended Frege.

Three papers in the volume are primarily devoted to equational calculi. JOHANNSEN defines calculi corresponding to the complexity worlds $\mathbf{ACC}^0[2]$ and $\mathbf{TC}^0$ (bounded depth threshold circuits). He checks the "correctness" of his definitions with the main test: proofs in these equational calculi can be efficiently simulated by the corresponding propositional proof systems $F_d(MOD_p)$ and FC. COOK considers quantified (but induction-free) versions QPV of the equational theory PV for the $\mathbf{P}$-world and a similar theory $QALV$ for the $\mathbf{NC}^1$-world. He shows that QPV is conservative over $QALV$ iff $QALV$ proves $\mathbf{P} = \mathbf{NC}^1$ iff $QALV$ proves its open induction formulae. PITT (work in progress) defines another equational theory for this $\mathbf{NC}^1$-world and argues that it might potentially turn out at least as convenient as the existing versions like $(Q)ALV$.

Three papers in the volume have their main focus on the arithmetical part of the triad T_C. POLLETT defines propositional proof systems corresponding to the first-order theories R_2^i of Bounded Arithmetic (the first member in this hierarchy, R_2^1 is the proof theory for the $\mathbf{NC}$-world). KAUFFMANN analyzes two known proofs of the classical result that the most important theory of Bounded Arithmetic S_2^1 is actually the proof theory for $\mathbf{P}$, i.e., that it Σ_1^b-defines exactly poly-time computable functions. She compares the efficiency of the resulting methods for extracting poly-time computable functions from S_2^1-proofs. BELLANTONI develops an alternative approach for identifying Bounded Arithmetic as a fragment of a natural hierarchy within PA. Instead of bounding the logical complexity of the formulae allowed in the induction *scheme*, this approach is based upon ranking *entire proofs* according to some ramification hierarchy.

Two remaining papers belong to adjacent areas. DOWNEY, FELLOWS and REGAN continue the study of the parameterized complexity theory and provide a descriptive logical characterization of the central complexity classes $W[t]$ in that area. WILLARD proposes interesting connections between some self-justifying systems and certain open questions about **NP**.

The volume under review adequately reflects current interests and trends in the area of Feasible Proofs, and many papers in it simply define the current state of art there. Its importance for everybody interested in this beautiful theory, be it the experienced researcher, a beginner or just a curious outsider, can hardly be overestimated.