An Equivalence between Second Order Bounded Domain Bounded Arithmetic and First Order Bounded Arithmetic

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Abstract

We introduce a bounded domain version $V_2^1(BD)$ of Buss's second order theory V_2^1 of bounded arithmetic and show that this version is equivalent to the first order theory S_3^1. More precisely, we construct two natural interpretations $V_2^1(BD) \rightsquigarrow S_3^1$ and $S_3^1 \rightsquigarrow V_2^1(BD)$ which are inverse to each other and preserve the syntactic structure of bounded formulae. As a corollary, for the bounded domain case we obtain Buss's result concerning $\Sigma_1^{1,b}$-expressibility in V_2^1 as a direct consequence of his main result for first order theories. Using only plain corollaries of the cut elimination theorem, we show that V_2^1 and $V_2^1(BD)$ prove the same $\Sigma^{1,b}$-formulae and the same *closed* $\forall^0\exists^0\Sigma^{1,b}$-formulae where $\forall^0, \exists^0$ stand for first order quantifiers. Combined with the above mentioned result this gives an alternative proof of Buss's characterization of $\Sigma_1^{1,b}$-definable in V_2^1 functions. All this readily extends to the case $V_k^i(BD)$ vs. S_{k+1}^i $(i, k \geq 1)$.

1. Introduction

The study of weak fragments of Peano Arithmetic has received much attention in last years partly because of close connection with important questions studied in the Computational Complexity. Originally the main target of this research was the theory $I\Delta_0$ and its subtheories. However, one considerable disadvantage of $I\Delta_0$ is that we can code in this theory finite sequences only of a fixed length prescribed in advance by the coding scheme. The smallest extension of $I\Delta_0$ where we can code sequences of length comparable with lengths of their elements (which is e.g. necessary for formalizing methamatematics in a natural way) is the theory $I\Delta_0 +$ "$\forall x\,(x^{\log x}$ exists)". This last theory (under the name S_2) and its fragments S_2^i, T_2^i obtained by further restricting the induction scheme were systematically studied in the seminal work of S.R.Buss [Bus86]. The main result of that work was a characterization of levels of the Meyer-Stockmeyer hierarchy in the Complexity theory (see e.g. [GJ79]) in terms of expressibility in certain fragments of S_2 [Bus86, theorems 3.1 and 5.1].

To capture higher levels of the complexity hierarchy such as $PSPACE$ or $EXPTIME$, Buss also introduced in [Bus86] second order theories U_2^1 and V_2^1 respectively. Clote and Takeuti [CT86] extended Buss's result concerning definability of $EXPTIME$ in V_2^1 to n-fold exponential time computable functions by utilizing some many-sorted theories of Bounded Arithmetic.

Theories U_2^1 and V_2^1 are included into hierarchies U_k^i and V_k^i which relate to first order hierarchies S_k^i and T_k^i more or less like the analytical hierarchy relates to the arithmetic hierarchy in the classical case. Hence it is not surprising that many definitions and proofs in second order theories are straightforward analogues of the corresponding definitions and proofs for first order theories.

But Bounded Arithmetic also gives an opportunity to *directly* interpret second order objects in a first order language. The antagonism "second order vs. first order" translates via this interpretation to "arbitrary integers vs. those "small" integers x for which $f(x)$ exists" where f is a rapidly growing function whose existence is not provable in Bounded Arithmetic.

This idea was first explored by Buss at the end of his book [Bus86] where he, based upon techniques of Solovay, Nelson and of Wilkie (see [Pud83]), sketched a proof that a bounded domain version of the second order bounded arithmetic can be interpreted in S_2 and hence (by a result of Wilkie and Nelson) is predicative in the sense of Nelson.

The main (informal) goal of this paper is to develop logical formalism which would reveal that second order theories of Bounded Arithmetic (at least those from the hierarchy V_k^i) not only *look* similar to their first order counterparts but basically *are* the same. Formally we do the following.

First we define[1] bounded domain versions $V_k^i(BD)$ of theories V_k^i. We

[1] The first attempt to do this was undertaken by Buss in [Bus86, §10.8]. This attempt however is not quite satisfactory since the theory $V_2^1(BD)$, as defined in [Bus86], strictly speaking is not a second order theory at all. For example, it contains the axiom $\forall x \forall \phi^a\,(x > a \supset \neg\phi^a(x))$ where ϕ^a is a second order variable of the sort associated with the term a. However, the instance of this axiom obtained by, say, substituting 0 for a is not intended to be a theorem of $V_2^1(BD)$.

will shortly discuss underlying principles of these theories in section 3. Here we only note that with the suggested version of the definition Buss's interpretation of $V_2(BD)$ in S_2 works out smoothly.

Then we step by step establish the main result of this paper. It says that there are two natural interpretations $\flat : V_2^1(BD) \rightsquigarrow S_3^1$ and $\sharp : S_3^1 \rightsquigarrow V_2^1(BD)$ such that $\flat$ maps closed $\Sigma_i^{1,b}$-formulae into closed Σ_i^b-formulae, $\sharp$ maps closed Σ_i^b-formulae into closed $\Sigma_i^{1,b}$-formulae and $\flat, \sharp$ are inverse to each other in the sense that $V_k^i(BD) \vdash A \equiv A^{\flat\sharp}$ and $S_{k+1}^i \vdash B \equiv B^{\sharp\flat}$ for any closed A, B in the appropriate language.

I would like to comment on the somewhat surprising fact that removing second order variables from, say, V_2^1 costs us exactly introducing the new symbol $\#_3$ from a computer scientist's point of view. It is well recognized in Theoretical Computer Science that when we change the representation of integers from binary to unary, the class $EXPTIME$ becomes *not* the class P but the class $TIME\left(2^{(\log n)^{O(1)}}\right)$. So the theory V_2^1 must correspond to a first order theory which captures this latter class. S_3^1 is exactly such a theory.

A straightforward application of this equivalence is a proof of the fact that functions which are $\Sigma_i^{1,b}$-definable in $V_2^i(BD)$ are exactly the functions from the i-th level of the exponential hierarchy. Aside from this equivalence, the proof uses only the corresponding result for S_3^i.

Lastly we compare the power of different bounded and unbounded domain versions of the theories V_2^i and show that all they prove the same $\Sigma^{1,b}$-formulae and the same closed $\forall^0\exists^0\Sigma^{1,b}$-formulae ($\forall^0, \exists^0$ stand for first order quantifiers). Combined with the above mentioned result this gives an alternative proof of Buss's characterization of $\Sigma_1^{1,b}$-definable in V_2^1 functions which uses only a plain consequence of the cut elimination theorem for Second Order Bounded Arithmetic.

The paper is organized as follows. In Section 2 we briefly remind the reader the necessary definitions and facts from [Bus86]. In Section 3 we define and discuss bounded domain counterparts of Buss's theories V_k^i. In two following sections we construct interpretations $\flat : V_2^1(BD) \rightsquigarrow S_3^1$ and $\sharp : S_3^1 \rightsquigarrow V_2^1(BD)$ respectively and establish their main properties (Lemmas 4.2, 5.4). In Section 6 we prove that these interpretations are inverse to each other (Lemmas 6.1, 6.2). In Section 7 we summarize in the Main Theorem (Theorem 7.1) facts established in previous sections and show that this theorem fairly easy allows us to translate results on Σ_i^b-definability in S_{k+1}^i to corresponding results on $\Sigma_i^{1,b}$-definability in $V_k^i(BD)$ (Theorem 7.2). In the next section 8 we show that all (bounded and unbounded domain) variants of V_k^i considered in [Bus86] and this paper prove the same $\Sigma^{1,b}$-formulae and the same closed $\forall^0\exists^0\Sigma^{1,b}$-formulae. We conclude the paper with a couple of remarks in Section 9.

Remark 1 *After the original version of this paper was disseminated, I learned that very similar results had been earlier proven by Takeuti* [Tak90, Tak91]*. The latter paper also contains a generalization to the case $U_k^i(BD)$ vs. R_{k+1}^i where R_k^i is the theory introduced in various forms by Allen* [All89]*, Clote* [Clo89] *and Takeuti* [Tak91].

2. Preliminaries

This section consists of primary definitions and facts concerning Bounded Arithmetic mostly borrowed from [Bus86]. We do not intend to give a self-contained account so some familiarity with [Bus86] is desirable.

Let the first order language L_k $(k > 1)$ with equality consist of the function symbols $0, S, +, \cdot, \lfloor \frac{1}{2}x \rfloor, |x|, x\#_2 y, \ldots, x\#_k y$ and of the predicate symbol $\leq$. Sometimes we will denote $\#_2$ by $\#$ and $\#_1$, for uniformity of notation, will always mean "$\cdot$". For a vector $\vec{x} = (x_1, \ldots, x_n)$ we will write $\bigwedge_{i=1}^n x_i \leq y$ and $\bigwedge_{i=1}^n x_i < y$ in the simplified form $\vec{x} \leq y$ and $\vec{x} < y$ respectively (where of course $x < y \rightleftharpoons x \leq y \wedge x \neq y$). The intended meaning of symbols $x\#_j y$ $(2 \leq j \leq k)$ is given recursively by

$$x\#_j y = 2^{|x|\#_{j-1}|y|} (2 \leq j \leq k). \tag{1}$$

The meaning of all other symbols is obvious.

Quantifiers of the form $(\forall x \leq t)$ and $(\exists x \leq t)$ are called *bounded*, quantifiers of the form $(\forall x \leq |t|)$ and $(\exists x \leq |t|)$ are called *sharply bounded* (here t is any term not involving x). A formula is *bounded* if all quantifiers in it are bounded. The hierarchy Σ_i^b of bounded formulae is defined by counting alternations of bounded quantifiers, ignoring the sharply bounded quantifiers. Also Σ^b stands for the set of all bounded formulae that is $\Sigma^b \rightleftharpoons \bigcup_{i \geq 0} \Sigma_i^b$.

Definition 2.1 The theory S_k^i is the first order theory in the language L_k with the following axioms:

1. 33 open axioms $BASIC_k$. This list coincides with Buss's list [Bus86, §2.2] with the difference that we generalize his axiom (13) $|x\#y| =$

$S(|x| \cdot |y|)$ to

$$|x \#_j y| = S(|x| \#_{j-1} |y|) \ (2 \leq j \leq k) \tag{2}$$

and add the new axiom

$$z < x \#_j y \equiv |z| < |x \#_j y| \ (2 \leq j \leq k). \tag{3}$$

2. $\Sigma_i^b - PIND$ that is the scheme

$$A(0) \wedge \forall x \left(A \left(\lfloor \frac{1}{2} x \rfloor \right) \supset A(x) \right) \supset \forall x A(x) \tag{4}$$

where $A \in \Sigma_i^b$.

Note that axioms (2) and (3) uniquely describe $\#_3, \ldots, \#_k$. Since these symbols are not used in bootstrapping S_k^i (see [Bus86, §2.4-2.5]), we do not care about including more their properties into $BASIC_k$.

Any new function symbol defined in S_k^i by a Σ_1^b-formula and any predicate symbol defined by a Δ_1^b-formula can be freely used in the formula A from (4). Among these are, for example, the following symbols:

- for each fixed $r > 0$ a function symbol $\langle x_1, \ldots, x_r \rangle$ which implements a one-to-one mapping $\mathbf{N}^r \longrightarrow \mathbf{N}$ and r unary symbols $\Pi_1^r, \ldots, \Pi_r^r$ representing the inverse mapping $\mathbf{N} \longrightarrow \mathbf{N}^r$. W.l.o.g. we may assume that
$$S_1^1 \vdash \langle x_1, \ldots, x_r \rangle \leq y \ \supset \ \vec{x} \leq y. \tag{5}$$
Also there exists a term $B^r(y)$ such that
$$S_1^1 \vdash \vec{x} \leq y \ \supset \ \langle x_1, \ldots, x_r \rangle \leq B^r(y), \tag{6}$$

- predicate symbol $Power2(x)$ and function symbol $max(x_1, \ldots, x_r)$ with obvious meaning,

- function symbol $BIT(i, x)$ which means "the bit of x in the 2^i position" and
$LSP(x, y)$ which means "the integer presented by y last bits of x".

All symbols above are already defined in S_1^1.

The main result proved by Buss concerns Σ_i^b-definability in the theory S_k^i. We state it here only for the case $i = 1$.

Proposition 2.2 ([Bus86, theorem 3.1 + theorem 5.1]) *Let $k \geq 2$.*

1. *for each function $f(n_1, \ldots, n_r) : \mathbf{N}^r \longrightarrow \mathbf{N}$ computable in time $|t(n_1, \ldots, n_r)|$ where t is a term of L_k (interpreted on $\mathbf{N}$ using (1)) there exists a Σ_1^b-formula $A(x_1, \ldots, x_r)$ in L_k such that:*
   - *(a)* $S_k^1 \vdash \exists y \leq t\ A(\vec{x}, y)$,
   - *(b)* $S_k^1 \vdash (A(\vec{x}, y) \wedge A(\vec{x}, z)\ \supset\ y = z)$,
   - *(c)* $\mathbf{N} \models A(\vec{n}, f(\vec{n}))$.
2. *conversely, suppose $S_k^1 \vdash \forall \vec{x} \exists y\ A(\vec{x}, y)$ where $A(\vec{x}, y)$ is a Σ_1^b-formula with all free variables displayed. Then there exists a term $t(\vec{x})$ in the language L_k, a Σ_1^b-formula $B(\vec{x}, y)$ and a function $f(\vec{n})$ computable in time $|t(\vec{n})|$ such that:*
   - *(a)* $S_k^1 \vdash B(\vec{x}, y) \supset A(\vec{x}, y)$,
   - *(b)* $S_k^1 \vdash B(\vec{x}, y) \wedge B(\vec{x}, z)\ \supset\ y = z$,
   - *(c)* $S_k^1 \vdash \exists y \leq t\ B(\vec{x}, y)$,
   - *(d) for all* $\vec{n}$, $\mathbf{N} \models B(\vec{n}, f(\vec{n}))$.

Let $\mathcal{L}_k$ be the second order language corresponding to L_k. It is well known that second order variables for functions can be easily simulated by second order variables for predicates (see e.g. [Bus86, Lemma 9.6]). So we define $\mathcal{L}_k$ to be the language obtained from L_k by augmenting it with second order variables $\{\alpha_i^r \mid i, r \in \mathbf{N};\ r \geq 1\}$ where r denotes the arity of the variable. The superscript r will be dropped whenever it can not create confusion.

Remark 2 Since we are going to talk of interpretations, it is important that we treat $\mathcal{L}_k$ as a many-sorted language (see e.g. [Tak75, §1.8]). Formally, this language has sorts $0, 1, 2, \ldots$ where 0 is reserved for the sort of first order variables and $r > 0$ is the sort of r-ary second order variables. It contains special predicate symbols $Value_r$ of type

$$(r+1; r, \underbrace{0, \ldots, 0}_{r \text{ times}})$$

such that $\alpha^r(x_1, \ldots, x_r) \rightleftharpoons Value_r(\alpha^r, x_1, \ldots, x_r)$. In particular, we are going to freely introduce function and predicate symbols containing second order variables as their arguments.

A term in the language L_k is called a *first order term.* A second order formula is *bounded* if all first order quantifiers in it are bounded. The hierarchy $\Sigma_i^{1,b}$ of bounded second order formulae is defined by counting second order quantifiers ignoring bounded first order quantifiers; $\Sigma^{1,b} \rightleftharpoons \bigcup_{i \geq 0} \Sigma_i^{1,b}$. Note that, like in the first order case, function symbols introduced by $\Sigma_1^{1,b}$-formulae and predicate symbols introduced by $\Delta_1^{1,b}$-formulae can be freely used in second order formulae since such usage does not bring bounded formulae out of the classes from the hierarchy $\Sigma_i^{1,b}$ they originally belonged to.

Definition 2.3 For a class Φ of formulae, Φ *comprehension axioms* (or $\Phi - CA$) is the following scheme:

$$\exists \alpha^r \forall\, x_1, \ldots, x_r\, \{\alpha^r(x_1, \ldots, x_r) \equiv A(x_1, \ldots, x_r)\}$$

where A is in Φ and does not contain α.

Definition 2.4 The second order theory $\widetilde{V}_k^i$ in the language $\mathcal{L}_k$ has the following axioms:

1. $BASIC_k$,
2. $\Sigma_i^{1,b} - IND$,
3. $\Sigma_0^{1,b} - CA$.

The theory $\mathring{V}_k^i$ differs from $\widetilde{V}_k^i$ in allowing a slightly stronger comprehension axiom.

Definition 2.5 $\mathring{V}_k^i$ is the second order theory in $\mathcal{L}_k$ with the following axioms:

1. $BASIC_k$,
2. $\Sigma_i^{1,b} - IND$,
3. $\Delta_1^{1,b} - CA$.

3. Bounded Domain Variants

In this section we define two bounded domain versions of the theories $\widetilde{V}^i_k$ and $\mathring{V}^i_k$ and prove their primary properties. Given our general goal, the idea is to describe in the second order language $\mathcal{L}_k$ exactly those properties which S^i_{k+1} can prove about sequences $Bit(0,x), Bit(1,x), \ldots, Bit(y,x), \ldots$ where $x \in \mathbf{N}$. And the first thing we discover is that the general CA-scheme becomes unsound and we should replace it by the following restricted scheme.

Definition 3.1 $\Phi - BCA$, Φ *bounded comprehension axioms* is the following scheme:

$$\exists \alpha^r \forall\, x_1, \ldots, x_r\, \{\alpha^r(x_1, \ldots, x_r) \equiv (A(x_1, \ldots, x_r) \wedge \vec{x} \leq t)\}$$

where A is in Φ and does not contain α and t is a first order term which does not contain variables from $\vec{x}$.

Definition 3.2 $V^i_k(bd)$ is the theory in the language $\mathcal{L}_k$ with the following set of axioms:

1. $BASIC_k$,
2. $\Sigma^{1,b}_i - IND$,
3. $\Sigma^{1,b}_0 - BCA$.

$V^i_k(bd)$ is a "neutral" theory which can neither prove nor disprove the existence of sequences with infinitely many ones. However we know that S^i_{k+1} can disprove this. So we must add a new axiom forbidding such sequences. This leads to the following

Definition 3.3 $V^i_k(BD)$ has the following axioms:

1. $BASIC_k$,
2. $\Sigma^{1,b}_i - IND$,
3. $\Sigma^{1,b}_0 - BCA$,

4. $\exists y \forall \vec{x}\, (\alpha(\vec{x}) \supset \vec{x} < y)$.

Now we are going to establish some simple properties of $V_k^i(BD)$. First we introduce the *second order equality* by

$$\alpha^r = \beta^r \rightleftharpoons \forall \vec{x}\, (\alpha^r(\vec{x}) \equiv \beta^r(\vec{x})) \tag{7}$$

and *bounded second order equality* by

$$\alpha^r \stackrel{y}{=} \beta^r \rightleftharpoons \forall \vec{x} \le y\, (\alpha^r(\vec{x}) \equiv \beta^r(\vec{x}))\,. \tag{8}$$

Note that whereas the unbounded second order equality is *not* allowed in axiom schemes 2, 3 of Definition 3.3, the bounded second order equality is given by the $\Sigma_0^{1,b}$-formula (8) and hence can be freely used there.

Now apply induction on y to the $\Sigma_0^{1,b}$-formula $A(\alpha, y, z) \rightleftharpoons \exists \vec{x} \le z\, (\neg \vec{x} < y \wedge \alpha(\vec{x}))$. $V_1^0(BD) \vdash \neg A(\alpha, z+1, z)$ and hence we have

$$\left.\begin{array}{l} V_1^0(BD) \vdash \forall \vec{x} \le z\, (\neg\alpha(\vec{x}))\ \vee\ \exists y > 0 \\ \qquad \left\{ \begin{array}{l} \exists \vec{x} \le z\, (\alpha(\vec{x}) \wedge max(x_1, \ldots, x_r) = y \dot{-} 1)\ \wedge \\ \forall \vec{x} \le z\, (\alpha(\vec{x}) \supset \vec{x} < y)\,. \end{array}\right. \end{array}\right\} \tag{9}$$

Let z be such that $\forall \vec{x}\, (\alpha(\vec{x}) \supset \vec{x} \le z)$; the existence of such z is assured by Definition 3.3 4. We obtain from (9):

$$\left.\begin{array}{l} V_1^0(BD) \vdash \forall \vec{x}\, (\neg\alpha(\vec{x}))\ \vee \\ \exists y > 0\ \{\exists \vec{x}\, (\alpha(\vec{x}) \wedge max(x_1, \ldots, x_r) = y \dot{-} 1)\ \wedge \\ \forall \vec{x}\, (\alpha(\vec{x}) \supset \vec{x} < y)\}\,. \end{array}\right\} \tag{10}$$

If we define now

$$\begin{array}{l} B(\alpha, y) \rightleftharpoons \forall \vec{x}\, (\alpha(\vec{x}) \supset \vec{x} < y)\ \wedge \\ \{y > 0 \supset \exists \vec{x}\, (\alpha(\vec{x}) \wedge max(x_1, \ldots, x_r) = y \dot{-} 1)\} \end{array}$$

then (10) implies $V_1^0(BD) \vdash \forall \alpha \exists y B(\alpha, y)$ and an easy logical analysis gives us $V_1^0(BD) \vdash B(\alpha, y) \wedge B(\alpha, y') \supset y = y'$. Hence we may introduce function symbols $\ell^r(\alpha^r)$ $(r \ge 1)$ of type $(1; r, 0)$[2] with defining axioms

$$\forall \vec{x} (\alpha(\vec{x}) \supset \vec{x} < \ell(\alpha)), \tag{11}$$

$$\ell(\alpha) > 0 \supset \exists \vec{x}\, (\alpha(\vec{x}) \wedge max(x_1, \ldots, x_r) = \ell(\alpha) \dot{-} 1)\,. \tag{12}$$

[2]the notation $(1; r, 0)$ reads as "1-ary function symbol with the argument of type r taking values in type 0" (see [Tak75, §1.8])

These symbols will be of crucial importance for the following; their intended meaning in the case $r = 1$ is "the length of α".

The next thing to do is to note that w.l.o.g. we may consider only unary second order variables which will highly simplify the notation. For this we implement one-to-one correspondence between $\mathbf{N}^r$ and $\mathbf{N}$ using $\Sigma_0^{1,b}$-definable function symbols

$$\langle x_1, \ldots, x_r \rangle,\ \Pi_1^r, \ldots, \Pi_r^r$$

from previous section. Using $\Sigma_0^{1,b} - BCA$ we prove that for each α^r there exists α^1 such that $\alpha^r(x_1, \ldots, x_r) \equiv \alpha^1(\langle x_1, \ldots, x_r \rangle)$ and vice versa (the bounding term t in Definition 3.1 is changed accordingly to (5), (6)). More formally, this leads to the following:

Theorem 3.4 *For a second order formula A define the formula $A^\star$ by replacing all its atomic subformulae of the form $\alpha_i^r(t_1, \ldots, t_r)$ by*

$$\alpha_{i,r}^1(\langle t_1, \ldots, t_r \rangle)$$

(variables $\alpha_{i,r}$ of sort 1 are assumed pairwise distinct). Then:

1. *for each $i \geq 0$, if A is $\Sigma_i^{1,b}$ then so is $A^\star$,*
2. *$\star$ defines an interpretation of $V_k^i(BD)$ in the theory obtained from $V_k^i(BD)$ by restricting the language $\mathcal{L}_k$ to variables of sorts 0 and 1 only,*
3. *$\star$ is inverse to the identical interpretation in the sense that for each formula A without free second order variables, $V_k^0(BD) \vdash A \equiv A^\star$.*

In view of Theorem 3.4, we will often restrict our considerations to the language which contains variables of types 0 and 1 only. Theorem 3.4 always allows us to extend corresponding results to the general case.

The reader may have observed that, unlike Section 2, in this section we have not used any special notation to indicate the strength of comprehension axioms involved in our theories. Note that it is open whether $\widetilde{V}_k^i \equiv \overset{\circ}{V}{}_k^i$ [Bus86, §10.8]. The next result shows that in bounded domain the answer to the analogous question is positive.

Theorem 3.5 *$V^i_k(bd)$ proves $\Delta^{1,b}_i - BCA$.*

Proof. Let $A(x)$ be a $\Delta^{1,b}_i$-formula. We prove by induction on y that

$$\exists\alpha\,\{\alpha(x) \equiv (A(x) \wedge x < y)\} \tag{13}$$

(note that this is a $\Sigma^{1,b}_i$-formula). The base $y = 0$ is obvious.

If we already have α_y such that $\alpha_y(x) \equiv (A(x) \wedge x < y)$ then in the case $\neg A(y)$ α_y also satisfies (13) with $y+1$ instead of y. If $A(y)$ then the required α_{y+1} is obtained from α_y by applying $\Sigma^{1,b}_0 - BCA$ to the formula $(\alpha(x) \vee x = y) \wedge x \leq y$. This completes the inductive step.

Substituting $t+1$ for y in (13) completes the proof of the theorem.■

Now, by $\Sigma^{1,b}_0 - BCA$, $V^0_k(bd) \vdash \exists\beta\forall x\,\{\beta(x) \equiv (\alpha(x) \wedge x \leq y)\}$. It is easy to see that such a β is unique and hence we have

$$V^0_k(bd) \vdash \exists!\beta\forall x\,\{\beta(x) \equiv (\alpha(x) \wedge x \leq y)\}\,.$$

So we may introduce in $V^0_k(bd)$ the function symbol $\alpha|_y$ of type (2;1,0,1) with the defining axiom $\alpha|_y(x) \equiv (\alpha(x) \wedge x \leq y)$. The following properties of $\alpha|_y$ are easy to check:

$$V^0_1(bd) \vdash \alpha|_y \stackrel{y}{=} \alpha, \tag{14}$$

$$V^0_1(BD) \vdash \ell\,(\alpha|_y) \leq y+1, \tag{15}$$

$$V^0_1(BD) \vdash \ell\,(\alpha) \leq y+1 \supset \alpha|_y = \alpha, \tag{16}$$

$$x = \ell(\alpha|_y) \text{ and } x \leq \ell(\alpha|_y) \text{ are } \Sigma^{1,b}_0 - \text{formulae (in } V^0_1(BD)). \tag{17}$$

Part 1 of the following theorem is a bounded version of [Tak75, Proposition 15.13] (see also [Bus86, Lemma 10.9]).

Theorem 3.6 *For any bounded formula $A(\alpha)$ there exists a first order term t such that:*

1. $V^0_k(bd) \vdash \alpha \stackrel{t}{=} \beta \supset A(\alpha) \equiv A(\beta)$,
2. $V^0_k(BD) \vdash \exists\alpha A(\alpha) \;\equiv\; \exists\alpha\,\{\ell(\alpha) \leq t+1 \wedge A(\alpha)\}$.

Proof. 1 Induction on complexity of A. If $A \equiv \alpha(t)$ then clearly $\alpha \stackrel{t}{=} \beta \supset \alpha(t) \equiv \beta(t)$. If $A \equiv B \circ C$ where $\circ$ is a Boolean connective, we let

$t_A \rightleftharpoons t_B + t_C$. In the case $A \equiv \exists x \leq sB(x)$ we set $t_A \rightleftharpoons t_B\,[s/x]$ (note that $S^1_k \vdash x \leq s \supset t \leq t\,[s/x]$ since first order terms are provably monotone). All other cases are trivial.

2 We assume that t is such that 1 holds. By (15), (14) and 1, $A(\alpha)$ implies $\ell(\alpha|_t) \leq t + 1 \wedge A(\alpha|_t)$. This gives us part 2 of the theorem.■

To conclude this section we mention the possibility to define in $V^1_k(BD)$ new function symbols (which may in general depend on second order variables) using "bounded" $\Delta^{1,b}_1$-abstracts.

Lemma 3.7 *Let $A(x_1, \ldots, x_r, \vec{y}, \vec{\alpha})$ be a $\Delta^{1,b}_1$-formula and $t(\vec{y}, \vec{z})$ be a first order term such that*

$$V^i_k(BD) \vdash A(\vec{x}, \vec{y}, \vec{\alpha}) \supset \vec{x} \leq t(\vec{y}, \ell(\vec{\alpha})). \tag{18}$$

Then, provided $i \geq 1$, we can introduce in $V^i_k(BD)$ the function symbol $f(\vec{y}, \vec{\alpha})$ taking values of sort r with the defining axiom

$$f(\vec{y}, \vec{\alpha})(x_1, \ldots, x_r) \equiv A(x_1, \ldots, x_r, \vec{y}, \vec{\alpha}). \tag{19}$$

Moreover, for each $j \leq 1$, any $\Sigma^{1,b}_j$-formula possibly containing f is equivalent in $V^i_k(BD)$ to a $\Sigma^{1,b}_j$-formula without f. Hence f can be freely used in $\Sigma^{1,b}_i - IND$, $\Delta^{1,b}_1 - BCA$ and repeating applications of this lemma.

Proof. By Theorem 3.5, we may apply $\Delta^{1,b}_1 - BCA$ to show

$$V^1_k(BD) \vdash \exists \beta \forall \vec{x} \left\{ \beta(\vec{x}) \equiv (A(\vec{x}, \vec{y}, \vec{\alpha}) \wedge \vec{x} \leq t(\vec{y}, \vec{z})) \right\}.$$

Substituting here $\ell(\vec{\alpha})$ for $\vec{z}$ and applying (18), we get

$$V^i_k(BD) \vdash \exists \beta \forall \vec{x} \left\{ \beta(\vec{x}) \equiv A(\vec{x}, \vec{y}, \vec{\alpha}) \right\}.$$

The uniqueness is obvious hence we may introduce the desired symbol f with the defining axiom (19). The second part of the lemma follows by standard argument (see e.g. [Bus86, theorem 2.4]).■

4. Interpretation $\flat$

In this section we construct an interpretation[3] $\flat : V^i_k(BD) \rightsquigarrow S^i_{k+1}$ and show that it preserves the syntactic structure of bounded formulae. The idea resembles the idea used by Buss in [Bus86, §10.8]. But instead of "pushing" first order variables down to the level defined by existence of the *double* exponent, we stay on the "logarithmic" level and insert instead the next smash function $\#_{k+1}$. It is the novelty which will allow us to show in subsequent sections that our interpretation is exact.

We define $\flat$ as follows. Both first and second order variables are interpreted by first order variables of the language L_{k+1}. First order quantifiers are relativized to the formula $\exists y\,(|y| = x)$. Second order quantifiers are not relativized. Symbols from the language L_k are interpreted by themselves. $\alpha(x)$ is interpreted as $Bit(x, \alpha) = 1$.

Before proving that so defined $\flat$ is really an interpretation with desired properties, let us establish the following useful auxiliary fact.

Lemma 4.1 *For each term $t(\vec{x})$ in the language L_k there exists a Σ^b_1-definable in S^1_{k+1} function symbol $t^+(\vec{x})$ such that $S^1_{k+1} \vdash |t^+(x_1, \ldots, x_r)| = t\,(|x_1|, \ldots, |x_r|)$.*

Proof. First we show that there exists a term $\bar{t}^+$ such that

$$S^1_{k+1} \vdash t\,(|\vec{x}|) \leq |\bar{t}^+(\vec{x})|. \tag{20}$$

This is easily done by induction on complexity of t; for the case of smash functions $\#_j$ $(1 \leq j \leq k)$ use (2). Now the desired symbol t^+ is defined by the following Σ^b_1-formula:

$$t^+(\vec{x}) = y \;\rightleftharpoons\; |y| = t(|\vec{x}|) \wedge (y = 0 \vee Power2(y))\,.$$

The existence condition follows from (20) and the fact

$$S^1_1 \vdash 0 < x \leq |z| \supset \exists y\,(|y| = x \wedge Power2(y))\,;$$

[3] In this paper we use the standard notion of interpretability (see e.g. [Sho67, Section 4.7]) extended in an obvious way to the many-sorted case. Bounded quantifiers $\exists x \leq t\ A(x)$ and $\forall x \leq t\ A(x)$ are simply treated as shorthand terms for $\exists x(x \leq t \wedge A(x))$ and $\forall x(x \leq t \supset A(x))$ respectively and do not require special care.

uniqueness easily follows from general properties of $Power2$ provable in S^1_1.■

Now we are ready to establish the main result of this section.

Lemma 4.2 *Let $A(x_1,\dots,x_n)$ be a formula in $\mathcal{L}_k$ with all free first order variables displayed. Then:*

1. *if A is $\Sigma^{1,b}_0$ then $A^\flat(|x_1|,\dots,|x_n|)$ is Δ^b_1 with respect to S^1_{k+1},*

2. *if A is $\Sigma^{1,b}_i$ $(i\geq 1)$ then $A^\flat(|x_1|,\dots,|x_n|)$ is equivalent in S^1_{k+1} to a Σ^b_i-formula,*

3. *if $i\geq 1$ then $\flat$ is an interpretation of $V^i_k(BD)$ in S^i_{k+1}.*

Proof. 1. Induction on complexity of A. The only nontrivial case to be considered is $A(\vec{x})\equiv \exists y\leq t(\vec{x})B(\vec{x},y)$. By definition of $\flat$, $S^1_k\vdash A^\flat(\vec{x})\equiv \exists z\left(|z|\leq t(\vec{x})\wedge B^\flat(\vec{x},|z|)\right)$. By Lemma 4.1,

$$S^1_{k+1}\vdash A^\flat(|\vec{x}|)\equiv \exists z\left(|z|\leq |t^+(\vec{x})|\wedge B^\flat(|\vec{x}|,|z|)\right)\equiv$$
$$\exists z\leq 2t^+(\vec{x})\left(|z|\leq |t^+(\vec{x})|\wedge B^\flat(|\vec{x}|,|z|)\right).$$

By inductive assumption, this gives us a Σ^b_1-representation of $A^\flat\left(|\vec{x}|\right)$. To get a Π^b_1-representation, we merely note that

$$S^1_{k+1}\vdash \exists z\leq u\ C(|z|)\ \equiv\ \exists y\leq |u|\forall z\leq u\left(|z|=y\supset C(|z|)\right). \qquad (21)$$

1 is proved.

As an intermediate step, we prove now the following weak form of 3:

Statement 4.3 *$\flat$ is an interpretation of $V^0_k(BD)$ in S^1_{k+1}.*

Proof of Statement 4.3. $BASIC_k$ axioms are interpreted by themselves. $\Sigma^{1,b}_0-IND$ translates to

$$A^\flat(0)\wedge\forall z\left\{A^\flat(|z|)\supset A^\flat\left(S(|z|)\right)\right\}\ \supset\ \forall z\ A^\flat(|z|)$$

which is equivalent in S^1_{k+1} to

$$A^\flat(0) \wedge \forall z \left(A^\flat\left(\left|\lfloor\frac{1}{2}z\rfloor\right|\right) \supset A^\flat\left(|z|\right)\right) \ \supset \ \forall z \ A^\flat(|z|).$$

By part 1 of Theorem 4.2, this is equivalent to an instance of $\Sigma^b_1 - PIND$.

$\Sigma^{1,b}_0 - BCA$ translates to

$$\exists\alpha\forall z \left(Bit(|z|,\alpha) = 1 \ \equiv \ \left(A^\flat(|z|) \wedge |z| \leq |t^+(\vec{x})|\right)\right)$$

which is easily seen to be equivalent in S^1_{k+1} to

$$\left.\begin{array}{l}\exists\alpha \leq 4t^+(\vec{x}) + 1 \ \forall z \leq t^+(\vec{x}) + 1 \ Bit(|z|,\alpha) = 1 \ \equiv \\ \left(A^\flat(|z|) \wedge |z| \leq |t^+(\vec{x})|\right).\end{array}\right\} \quad (22)$$

Now, it is easy to prove by $\Sigma^b_1 - PIND$ on u that

$$\exists\alpha \leq 4u + 1 \ \forall z \leq u + 1 \left(Bit(|z|,\alpha) = 1 \ \equiv \ \left(A^\flat(|z|) \wedge |z| \leq |u|\right)\right)$$

(note this is Σ^b_1 by (21)). (22) follows.

Lastly, Definition 3.3 4 translates to $\exists y\forall z\,(Bit(|z|,\alpha) = 1 \supset |z| < |y|)$ which follows in S^1_1 from $Bit(x,\alpha) = 1 \supset x < |\alpha|$. This completes the proof of Statement 4.3.■

Now we continue the proof of Lemma 4.2.

2 By Statement 4.3, $\ell^\flat$ is properly defined in S^1_{k+1} and it follows from $(11)^\flat$,$(12)^\flat$ that $S^1_{k+1} \vdash \ell^\flat(\alpha) \equiv |\alpha|$. We apply again induction on complexity of A. The case $A \equiv \exists y \leq tB$ is considered as in the proof of part 1. Let $A(\vec{x}) \equiv \exists\alpha B(\alpha,\vec{x})$. By Theorem 3.6 2 and Statement 4.3,

$$S^1_{k+1} \vdash A^\flat\left(|\vec{x}|\right) \equiv \exists\alpha\left(|\alpha| \leq t\left(|\vec{x}|\right) + 1 \wedge B^\flat\left(\alpha,|\vec{x}|\right)\right). \quad (23)$$

for some first order term t. Since, by Lemma 4.1, $S^1_{k+1} \vdash |\alpha| \leq t\left(|\vec{x}|\right) + 1 \supset \alpha \leq 4t^+(\vec{x}) + 1$, the quantifier $\exists\alpha$ in (23) is bounded and 2 is proved.

3 The $\flat$-image of the last remaining axiom scheme $\Sigma^{1,b}_i - IND$ is provable in S^i_{k+1} by 2 and the same argument which was used in the proof of Statement 4.3 to deal with $\Sigma^{1,b}_0 - IND$.■

5. Interpretation $\sharp$

In this section we construct an interpretation $\sharp : S^i_{k+1} \rightsquigarrow V^i_k(BD)$ $(i, k \geq 1)$ which interprets first order variables of the language L_{k+1} by second order variables of $\mathcal{L}_k$. The universe of $\sharp$ consists of all α that is first order quantifiers become second order unrelativized quantifiers. Recall from Section 4 that bounded quantifiers are simply treated as shorthand terms. In order to complete the definiton of $\sharp$, we are only left to interpret nonlogical symbols of the language L_{k+1}.

We divide them into two groups. The first group consists of

$$\left\{0, \lfloor\frac{1}{2}x\rfloor, S, +\right\}.$$

These symbols will be called *local* and will be $\Delta^{1,b}_1$-defined using standard bit-operating computational algorithms. The remaining symbols

$$\{|x|, x\#_2 y, \ldots, x\#_k y, \leq\}$$

are *global*; their definitions will be based on the symbol ℓ.

We start with local symbols.

Let us apply Lemma 3.7 to $A(x) \rightleftharpoons 0 = 1,\ t \rightleftharpoons 0$. We will get a second order constant 0 with the defining axiom

$$0(x) \ \equiv \ 0 = 1.$$

Let now $A(x, \alpha) \rightleftharpoons \alpha(x+1)$ and $t(z) \rightleftharpoons z$. Clearly, (11) implies $V^0_1(BD) \vdash \alpha(x+1) \supset x \leq \ell(\alpha)$ hence we may introduce by Lemma 3.7 the symbol $\lfloor\frac{1}{2}\alpha\rfloor$ of type (1;1,1) with the defining axiom

$$\lfloor\frac{1}{2}\alpha\rfloor(x) \equiv \alpha(x+1).$$

It is easy to see that $V^0_1(BD) \vdash (\alpha(x) \equiv \exists y < x\ \neg\alpha(y)) \supset x \leq \ell(\alpha)$. Hence we may define $S\alpha$ such that

$$S\alpha(x) \equiv \{\alpha(x) \equiv \exists y < x\ \neg\alpha(y)\}.$$

Addition is defined by the "school" algorithm. Namely, what we would like to have is

$$\left.\begin{array}{l}(\alpha_1 + \alpha_2)(x) \equiv \exists\beta \\ \qquad\left\{\begin{array}{l}\neg\beta(0)\ \wedge \\ \forall 0 < y \leq x \\ \{\beta(y) \equiv T_2\left(\alpha_1(y \dot{-} 1), \alpha_2(y \dot{-} 1), \beta(y \dot{-} 1)\right)\} \wedge \\ \alpha_1(x) \oplus \alpha_2(x) \oplus \beta(x).\end{array}\right.\end{array}\right\} \quad (24)$$

Here $T_2(x,y,z) \rightleftharpoons (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ is the *threshold* function and $x \oplus y \rightleftharpoons x \not\equiv y$ is *addition mod* 2. The idea is that $\beta(y)$ represents the carry bit to the y's position.

To justify this definition denote

$$\neg\beta(0) \wedge \forall 0 < y \leq x \, \{\beta(y) \equiv T_2\,(\alpha_1(y \dot- 1), \alpha_2(y \dot- 1), \beta(y \dot- 1))\}$$

by $Carry(\alpha_1, \alpha_2, \beta, x)$. Then $\Sigma_1^{1,b} - IND$ proves

$$\exists\beta\ Carry(\alpha_1, \alpha_2, \beta, x)$$

and

$$\forall\beta, \gamma \left(Carry(\alpha_1, \alpha_2, \beta, x) \wedge Carry(\alpha_1, \alpha_2, \gamma, x) \ \supset\ \beta \stackrel{x}{=} \gamma\right)$$

which implies that the right-hand side of (24) actually belongs to $\Delta_1^{1,b}$. Moreover, $\neg\alpha_1(y) \wedge \neg\alpha_2(y)$ implies $\neg\beta(y+1)$ hence if in addition $\neg\alpha_1(y+1) \wedge \neg\alpha_2(y+1)$, then $\neg(\alpha_1 + \alpha_2)(x+1)$. All this shows that Lemma 3.7 is applicable (with $t(z_1, z_2) \rightleftharpoons z_1 + z_2$) and the definition (24) of $\alpha_1 + \alpha_2$ is justified.

Before defining the multiplication it is convenient to introduce two auxiliary symbols. The *projection* $\gamma^2|_y$ is the function symbol of the type (2;2,0,1) defined by

$$\gamma^2|_y(x) \equiv \gamma^2(x, y).$$

Define also "the easy case of multiplication" $2^y * \beta$ of type (2;0,1,1) by

$$2^y * \beta(x) \equiv x \geq y \wedge \beta(x \dot- y).$$

Now we define the multiplication by

$$\left.\begin{array}{l}
(\alpha \cdot \beta)(x) \equiv \exists\gamma^2 \\
\quad \left\{\begin{array}{l}
\gamma^2|_0 \stackrel{x}{=} 0 \ \wedge\ \forall 0 < y \leq x+1 \\
\left(\neg\alpha(y) \supset \gamma^2|_y \stackrel{x}{=} \gamma^2|_{y \dot- 1} \wedge\right. \\
\left.\alpha(y) \supset \gamma^2|_y \stackrel{x}{=} \gamma^2|_{y \dot- 1} + 2^{y \dot- 1} * \beta\right) \ \wedge \\
\gamma^2(x, x+1).
\end{array}\right.
\end{array}\right\} \quad (25)$$

The intuitive idea behind (25) is that the last $(x+1)$ bits of $\gamma^2|_y$ represent the last $(x+1)$ bits of the product $LSP(\alpha, y) \cdot \beta$.

To justify this definition we, similarly to the case of addition, define

$$Table(\alpha,\beta,\gamma^2,x) \rightleftharpoons \gamma^2|_0 \overset{x}{=} 0 \ \wedge\ \forall 0 < y \le x+1 \begin{cases} \neg\alpha(y) \supset \gamma^2|_y \overset{x}{=} \gamma^2|_{y \dot{-} 1} \ \wedge \\ \alpha(y) \supset \gamma^2|_y \overset{x}{=} \gamma^2|_{y \dot{-} 1} + 2^{y \dot{-} 1} * \beta \end{cases}$$

and prove that

$$V_1^1(BD) \vdash \exists\gamma^2\ Table(\alpha,\beta,\gamma^2,x)$$

and

$$V_1^1(BD) \vdash Table(\alpha,\beta,\gamma^2,x) \wedge Table(\alpha,\beta,\delta^2,x)\ \supset\ \forall y \le x+1 \left(\gamma^2|_y \overset{x}{=} \delta^2|_y\right).$$

This gives us that the right-hand side of (25) belongs to $\Delta_1^{1,b}$. Put $t(z_1,z_2) \rightleftharpoons z_1 + z_2$. By $\Sigma_0^{1,b} - IND$ on y we prove the following:

$$Table(\alpha,\beta,\gamma^2,x) \wedge \forall u \le x+1\,(\beta(u) \supset u < z_2) \wedge \gamma^2(x,y)\ \supset\ x \le y + z_2$$

and then

$$\begin{array}{l} Table(\alpha,\beta,\gamma^2,x) \wedge \forall u \le x+1\,(\beta(u) \supset u < z_2) \wedge \\ \forall u \le x+1\,(\alpha(u) \supset u < z_1) \wedge \gamma^2(x,y)\ \supset\ x \le z_1 + z_2. \end{array}$$

This clearly implies (18) for multiplication.

Let us $\Delta_1^{1,b}$-define two more symbols $id(y)$ and $exp(y)$, of type (1;0,1) each:

$$id(y)(x) \quad\equiv\quad Bit(x,y) = 1, \tag{26}$$

$$exp(y)(x) \quad\equiv\quad x = y. \tag{27}$$

Now we are in position to interpret remaining (global) symbols of the language L_{k+1}. We do this as follows:

$$\begin{array}{rcl} |\alpha| & \rightleftharpoons & id(\ell(\alpha)), \\ \alpha\#_j\beta & \rightleftharpoons & exp\,(\ell(\alpha)\#_{j-1}\ell(\beta))\,,\ (2 \le j \le k+1) \\ \alpha \le \beta & \rightleftharpoons & \alpha = \beta \vee \exists x\,(\neg\alpha(x) \wedge \beta(x) \wedge \forall y > x\,\{\alpha(y) \equiv \beta(y)\})\,. \end{array} \tag{28}$$

This completes the definition of a translation from the set of all (first order) formulae in the language L_{k+1} to the set of (second order) formulae in the language $\mathcal{L}_k$. We denote this translation by $\sharp$. The rest of the section is devoted to proving that $\sharp$ is actually an interpretation.

First we indicate several easy properties of $\sharp$.

Lemma 5.1 (soundness of id**)** *For each first order term* $t(\vec{x})$ *in* L_k (!) *we have*

$$V_k^1(BD) \vdash id(t(\vec{x})) = t^\sharp(id(\vec{x})).$$

Proof. It suffices to check this for function symbols. This is done straightforwardly. For example, for the case of multiplication we prove by induction on y that

$$Table\left(id(x_1), id(x_2), \gamma^2, j\right) \wedge y \leq j+1 \supset id\left(LSP(x_1,y)\cdot x_2\right) \overset{j}{=} \gamma^2|_y.$$

(2),(3) imply $S_k^1 \vdash Bit\,(z, x\#_j y) = 1 \equiv z = |x|\#_{j-1}|y|$ which, along with (26),(27) gives us

$$id(x\#_j y) = exp\left(|x|_{j-1}|y|\right) = exp\left(\ell(id(x))\#_{j-1}\ell(id(y))\right) = id(x)\#_j id(y)$$

etc.∎

The following lemma is in a sense dual to Lemma 4.1.

Lemma 5.2 *For each term* $t(\vec{x})$ *in the language* L_{k+1} *there exists a term* $t^-(\vec{x})$ *in the language* L_k *such that* $V_k^1(BD) \vdash \ell\left(t^\sharp(\vec{\alpha})\right) \leq t^-(\ell(\vec{\alpha}))$.

Proof. Straightforward (note that for local function symbols f the desired upper bound on $\ell\left(f^\sharp(\vec{\alpha})\right)$ was established in the process of justifying their definitions).∎

Lemma 5.3 *For each term* $t(x_1,\ldots,x_r)$ *in the language* L_{k+1} *there exists a* $\Delta_1^{1,b}$*-formula* $A_f(\alpha_1,\ldots,\alpha_r,z_1,\ldots,z_r,y)$ *such that*

$$V_k^1(BD) \vdash t^\sharp(\alpha_1,\ldots,\alpha_r)(y) \equiv A_t(\alpha_1,\ldots,\alpha_r,\ell(\alpha_1),\ldots,\ell(\alpha_r),y).$$

Proof. First check this for function symbols.

The claim of our lemma is obvious for local symbols; we merely take their defining axioms as corresponding A_f's (z-variables are dummy). Also let $A_{|x|}(\alpha,z,y) \rightleftharpoons Bit(y,z) = 1$ and $A_{x_1\#_j x_2}(\alpha_1,\alpha_2,z_1,z_2,y) \rightleftharpoons y = z_1\#_{j-1}z_2$.

Now if $t(\vec{x}) \equiv f(\vec{s}(\vec{x}))$ then

$$f^\sharp\left(\vec{s}^\sharp(\vec{\alpha})\right)(y) \equiv A_f\left(\vec{s}^\sharp(\vec{\alpha}), \ell\left(\vec{s}^\sharp(\vec{\alpha})\right), y\right) \equiv \text{ (by Lemma 5.2 and (16))}$$
$$\exists \vec{u} \leq \vec{s}^-(\ell(\vec{\alpha}))\left(A_f\left(\vec{s}^\sharp(\vec{\alpha}), \vec{u}, y\right) \wedge \vec{u} = \ell\left(\vec{s}^\sharp(\vec{\alpha})\left|_{\vec{s}^-(\ell(\vec{\alpha}))}\right.\right)\right).$$

This is $\Delta_1^{1,b}$ in $\vec{\alpha}, \ell(\vec{\alpha}), y$ by (17) and inductive assumption.■

Now we are in position to establish the main result of this section.

Lemma 5.4 *1. Let $A(x_1, \ldots, x_r)$ be a formula in the language L_{k+1} with all its free variables displayed. Then there exists a formula $\bar{A}^\sharp(\alpha_1, \ldots, \alpha_r, z_1, \ldots, z_r)$ such that $V_k^1(BD) \vdash A^\sharp(\vec{\alpha}) \equiv \bar{A}^\sharp(\vec{\alpha}, \ell(\vec{\alpha}))$ and:*

(a) if A is Σ_0^b then $\bar{A}^\sharp$ is $\Delta_1^{1,b}$ with respect to $V_k^1(BD)$,

(b) if A is Σ_i^b $(i \geq 1)$ then $\bar{A}^\sharp$ is $\Sigma_i^{1,b}$.

2. $\sharp$ is an interpretation of S_{k+1}^i in $V_k^i(BD)$ provided $i \geq 1$.

Proof. 1 We construct $\bar{A}^\sharp$ by induction on complexity of A. First we set $\overline{t = s}^\sharp \rightleftharpoons t^\sharp(\vec{\alpha}) \stackrel{z}{=} s^\sharp(\vec{\alpha})$ where $z \rightleftharpoons t^-(\ell(\vec{\alpha})) + s^-(\ell(\vec{\alpha}))$. We have $\overline{t = s}^\sharp \equiv \left(t^\sharp = s^\sharp\right)$ by Lemma 5.2, (16), (14) and it is in $\Delta_1^{1,b}$ by Lemma 5.3. $\overline{t \leq s}^\sharp$ is defined similarly. If $A \equiv B \circ C$ where $\circ$ is a Boolean connective then $\bar{A}^\sharp \rightleftharpoons \bar{B}^\sharp \circ \bar{C}^\sharp$.

Consider the case $A \equiv \exists x \leq |t(\vec{y})|\ B(x, \vec{y})$. We have

$$A^\sharp \equiv \exists \alpha \leq id\left(\ell(t^\sharp(\vec{\beta}))\right) B^\sharp(\alpha, \vec{\beta}) \equiv \exists \alpha \leq id\left(\ell(t^\sharp(\vec{\beta}))\right) \bar{B}^\sharp(\alpha, \vec{\beta}, \ell(\alpha), \ell(\vec{\beta})).$$

Let $\bar{A}^\sharp \rightleftharpoons \exists y \leq \ell(t^\sharp(\vec{\beta}))\ \bar{B}^\sharp(id(y), \vec{\beta}, |y|, \ell(\vec{\beta}))$. Like in the proof of Lemma 5.3 note that this is equivalent to

$$\exists y \leq t^-(\ell(\vec{\beta}))\left(y \leq \ell\left(t^\sharp(\vec{\beta})\left|_{t^-(\ell(\vec{\beta}))}\right.\right) \wedge \bar{B}^\sharp(id(y), \vec{\beta}, |y|, \ell(\vec{\beta}))\right)$$

which has the same complexity as $\bar{B}^\sharp$. The equivalence $\bar{A}^\sharp \equiv A^\sharp$ follows from the fact

$$V_1^1(BD) \vdash \alpha \leq id(x) \equiv \exists y \leq x(id(y) = \alpha). \tag{29}$$

To prove (29), we show by induction on x that $\exists y(|y| \le |x|+1)\left(id(y) \overset{|x|}{=} \alpha\right)$ and note that $V_1^0(BD) \vdash \alpha \le id(x) \wedge |y| \le |x|+1 \wedge id(y) \overset{|x|}{=} \alpha \supset id(y) = \alpha$.

Consider the last remaining case $A \equiv \exists x \le t(\vec{y})\ B(x, \vec{y})$. We have $A^\sharp \equiv \exists \alpha \le t^\sharp(\vec{\beta})\ \bar{B}^\sharp(\alpha, \vec{\beta}, \ell(\alpha), \ell(\vec{\beta}))$. The problem is to eliminate $\ell(\alpha)$. Since ℓ is monotone, we can use the same trick as before and let

$$\bar{A}^\sharp \rightleftharpoons \exists\alpha\exists z \le t^-(\ell(\vec{\beta}))\left(\overline{\alpha \le t^\sharp(\vec{\beta})} \wedge z = \ell\left(\alpha\Big|_{t^-(\ell(\vec{\beta}))}\right) \wedge \bar{B}^\sharp(\alpha, \vec{\beta}, z, \ell(\vec{\beta}))\right).$$

The first conjunctive term is in $\Delta_1^{1,b}$ and the second is in $\Sigma_0^{1,b}$ by (17).

The definition of the formula $\bar{A}^\sharp$ is completed. The syntactic analysis of its structure presented in the course of construction proves part 1 of Lemma 5.4.

2 First we will show that $V_k^1(BD)$ is strong enough to prove primary properties $(BASIC_k)^\sharp$ of function symbols and relations introduced in this section. This is straightforward for most axioms so we give only proof sketches for a few illustrating examples. We refer to the $\sharp$-image of the i-th axiom in the $BASIC$ list [Bus86, §2.2] as to $(\mathbf{Bi})^\sharp$.

$(\mathbf{B1})^\sharp$ $(\beta \le \alpha \supset \beta \le S\alpha)$. Applying $\Sigma_0^{1,b} - IND$ on x to the formula $\alpha(x)$, define the function symbol $\ell^*(\alpha)$ with the defining axiom $\neg\alpha\,(\ell^*(\alpha)) \wedge \forall x < \ell^*(\alpha)\ \alpha(x)$. Then $S\alpha$ is alternatively described as

$$S\alpha(x) \equiv \begin{cases} 0 = 1, & x < \ell^*(\alpha) \\ 0 = 0, & x = \ell^*(\alpha) \\ \alpha(x), & x > \ell^*(\alpha). \end{cases}$$

Now $(\mathbf{B1})^\sharp$ easily follows from the definition (28) using case analysis.

$(\mathbf{B5})^\sharp$ $(\alpha \ne 0 \supset 2 \cdot \alpha \ne 0)$. First we prove

$$V_1^1(BD) \vdash 2^y * \beta = exp(y) \cdot \beta. \tag{30}$$

This is done by applying $\Sigma_1^{1,b} - IND$ on u to the formula

$$Table\left(exp(y), \beta, \gamma^2, x\right) \wedge 0 < u \le x+1 \supset \gamma^2|_u \overset{x}{=} \begin{cases} 0, & u \le y \\ 2^y * \beta, & y < u \le x+1. \end{cases}$$

Now $(\mathbf{B5})^\sharp$ is implied by (30) with $y = 1$.

$(\mathbf{B6})^\sharp$ $(\alpha \le \beta \vee \beta \le \alpha)$. $\Delta_1^{1,b}$-define the function symbol $\alpha \oplus \beta$ such that

$$(\alpha \oplus \beta)(x) \equiv (\alpha(x) \oplus \beta(x)).$$

$\alpha \neq \beta$ implies $\alpha \oplus \beta \neq 0$ and $\ell(\alpha \oplus \beta) > 0$. Now, $\alpha(x)$ and $\beta(x)$ coincide for $x \geq \ell(\alpha \oplus \beta)$ and differ at $x = \ell(\alpha \oplus \beta) \dot{-} 1$. $(\mathbf{B6})^\sharp$ follows.

$(\mathbf{B10})^\sharp$ $(\alpha \neq 0 \supset |2 \cdot \alpha| = S(|\alpha|))$. By (30), $V_1^1(BD) \vdash \alpha \neq 0 \supset \ell(2 \cdot \alpha) = S(\ell(\alpha))$. Applying id and using Lemma 5.1, we get $V_1^1(BD) \vdash \alpha \neq 0 \supset |2 \cdot \alpha| = id(S(\ell(\alpha))) = S(id(\ell(\alpha))) = S(|\alpha|)$.

$(\mathbf{B12})^\sharp$ $(\alpha \leq \beta \supset |\alpha| \leq |\beta|)$. $V_1^1(BD)$ proves that both ℓ and id are monotonic.

$(2)^\sharp$ $V_k^1(BD) \vdash |\alpha \#_j \beta| = id\,(\ell\,(exp\,(\ell(\alpha)\#_{j-1}\ell(\beta)))) =$
$id\,(S\,(\ell(\alpha)\#_{j-1}\ell(\beta))) =$ (by Lemma 5.1) $S\,(id(\ell(\alpha))\#_{j-1}id(\ell(\beta))) =$
$S\,(|\alpha|_{j-1}|\beta|)$.

$(\mathbf{B17})^\sharp$ $(|\alpha| = |\beta| \supset \alpha\#\gamma = \beta\#\gamma)$. V_2^1 readily proves $id(x) = id(y) \supset x = y$ and hence $|\alpha| = |\beta| \supset \ell(\alpha) = \ell(\beta)$. But the definition of $\alpha\#\gamma$ depends only on $\ell(\alpha), \ell(\gamma)$.

$(\mathbf{B19})^\sharp$ $(\alpha_1 \leq \alpha_1 + \alpha_2)$. Prove by induction on y that

$$Carry(\alpha_1, \alpha_2, \beta, x) \wedge 0 < y \leq x \wedge \neg\beta(y) \ \supset\ \alpha_1|_{y \dot{-} 1} \leq (\alpha_1 + \alpha_2)|_{y \dot{-} 1}$$

and substitute $x = y = \ell(\alpha_1) + \ell(\alpha_2)$.

$(\mathbf{B21})^\sharp$ $(\alpha + \beta = \beta + \alpha)$. The definition of addition is symmetric.

$(\mathbf{B24})^\sharp$ $((\alpha + \beta) + \gamma = \alpha + (\beta + \gamma))$. Define the ternary *symmetric* function symbol $(\alpha + \beta + \gamma)$ in the obvious way. Then prove separately $(\alpha + \beta) + \gamma = (\alpha + \beta + \gamma)$ and $\alpha + (\beta + \gamma) = (\alpha + \beta + \gamma)$ by joint induction on all natural relations between $\alpha(x), \beta(x), \gamma(x), (\alpha + \beta)(x), (\alpha + \beta + \gamma)(x)$ etc. and all carry bits involved in the computations.

$(\mathbf{B29})^\sharp$ $(\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma))$. First we prove $2^y * (\beta + \gamma) = (2^y * \beta) + (2^y * \gamma)$. Then we get $(\mathbf{B29})^\sharp$ from this and $(\mathbf{B21})^\sharp$,$(\mathbf{B24})^\sharp$ by applying induction on y to the formula

$$\begin{gathered} Table(\alpha, \beta + \gamma, \delta^2, x) \wedge Table(\alpha, \beta, \delta_1^2, x) \wedge \\ Table(\alpha, \gamma, \delta_2^2, x) \wedge 0 < y \leq x + 1 \ \supset \\ \delta^2|_y \stackrel{x}{=} \delta_1^2|_y + \delta_2^2|_y. \end{gathered}$$

$(\mathbf{B30})^\sharp$ $(\alpha \geq 1 \ \supset\ \alpha \cdot \beta \leq \alpha\gamma \equiv \beta \leq \gamma)$. In view of $(\mathbf{B6})^\sharp$,$(\mathbf{B7})^\sharp$ we only have to check $V_1^1(BD) \vdash \alpha \neq 0 \wedge \beta < \gamma \ \supset\ \alpha \cdot \beta < \alpha \cdot \gamma$. This is done by induction like in $(\mathbf{B5})^\sharp$, $(\mathbf{B29})^\sharp$.

Now we check $V_k^i(BD) \vdash \left(\Sigma_i^b - PIND\right)^\sharp$. By already proven part 1, it suffices to show that

$$\left.\begin{array}{l} V_k^i(BD) \vdash \\ \left[B(0,0) \wedge \forall\alpha \left(B\left(\lfloor\frac{1}{2}\alpha\rfloor, \ell\left(\lfloor\frac{1}{2}\alpha\rfloor\right)\right) \supset B(\alpha, \ell(\alpha))\right)\right] \supset \\ \forall\alpha\ B(\alpha, \ell(\alpha)) \end{array}\right\} \quad (31)$$

where $B(\alpha, z)$ is in $\Sigma_i^{1,b}$. Let us introduce the function symbol $2^{-y}\alpha$ by

$$2^{-y}\alpha(x) \equiv \alpha(x+y).$$

Now induction on y applied to the $\Sigma_i^{1,b}$-formula

$$C(\alpha, z, y) \rightleftharpoons B\left(2^{-(z \dot{-} y)}\alpha, y\right)$$

implies (31) after substituting $z = y = \ell(\alpha)$.

The proof of Lemma 5.4 is completed.■

6. $\flat$ and $\sharp$ are inverse to each other

The primary purpose of this section is to justify its title by proving the following two statements:

Lemma 6.1 *Let $A(\vec{x})$ be any formula in $\mathcal{L}_k$ with all free first order variables displayed. Then $V_k^1(BD) \vdash A^{\flat\sharp}(id(\vec{x})) \equiv A(\vec{x})$. In particular, if A does not contain free first order variables, $V_k^1(BD) \vdash A^{\flat\sharp} \equiv A$.*

Lemma 6.2 *Let A be any formula in L_{k+1}. Then $S_{k+1}^1 \vdash A^{\sharp\flat} \equiv A$.*

Proof of Lemma 6.1. Induction on complexity of A. If $A \equiv t(\vec{x}) \circ s(\vec{x})$ where $\circ$ is $=$ or $\leq$ then $A^{\flat\sharp} \equiv t^{\sharp}(\vec{\alpha}) \circ s^{\sharp}(\vec{\alpha})$. By Lemma 5.1, $V_k^1(BD) \vdash A^{\flat\sharp}(id(\vec{x})) \equiv id(t(\vec{x})) \circ id(s(\vec{x})) \equiv A(\vec{x})$ since $V_1^1(BD)$ readily proves $x \circ y \equiv id(x) \circ id(y)$.

Assume now that $A \equiv \alpha(t(\vec{x}))$. Then $A^{\flat\sharp} \equiv Bit^{\sharp}\left(t^{\sharp}(\vec{x}), \alpha\right) = 1$ and $A^{\flat\sharp}(id(\vec{x})) \equiv Bit^{\sharp}\left(id(t(\vec{x})), \alpha\right) = 1$. So we only have to understand that

$$V_1^1(BD) \vdash Bit^{\sharp}(id(y), \alpha) = 1 \equiv \alpha(y). \quad (32)$$

The easiest way of doing this is to translate via $\sharp$ identities $Bit(0, 2x) = 0$, $Bit(0, 2x+1) = 1$, $Bit(y+1, x) = Bit(y, \lfloor \frac{x}{2} \rfloor)$ provable in S^1_1 to identities in $V^1_1(BD)$. Having these we easily establish by induction on z that $V^1_1(BD) \vdash Bit^\sharp \left(id(z), 2^{-(y \dot{-} z)}\alpha \right) = 1 \equiv \alpha(y)$ which (with $z = y$) gives us (32).

Suppose $A \equiv \exists x\ B(x)$. Then $A^\flat \equiv \exists z\ B^\flat(|z|)$ and $A^{\flat\sharp} \equiv \exists \alpha\ B^{\flat\sharp}(|\alpha|)$. To apply the inductive hypothesis we only have to show that $V^1_k(BD) \vdash \exists \alpha\ C(|\alpha|) \equiv \exists x\ C(id(x))$. It is clear in one direction since $|\alpha| \equiv id(\ell(\alpha))$. In another direction this follows from the identity $V^1_1(BD) \vdash id(x) = |exp(x)|$.

All remaining cases are trivial.■

The following immediate corollary is interesting in its own right.

Corollary 6.3 *For each $i \geq 1$ the interpretation $\flat$ is exact. In other words, for each closed formula A in $\mathcal{L}_k$, $V^1_k(BD) \vdash A$ iff $S^1_{k+1} \vdash A^\flat$.*

Proof of Lemma 6.2. Induction on complexity of A. The only nontrivial case is when A is atomic and it suffices to show for that case that

$$S^1_{k+1} \vdash t^{\sharp\flat}(\vec{x}) = t(\vec{x}) \tag{33}$$

for any term t in L_{k+1} and $S^1_{k+1} \vdash x \leq^{\sharp\flat} y \equiv x \leq y$. It is sufficient to establish (33) only for the case when t is a function symbol of the language L_{k+1}. The easiest way to do this is to note that the system $\left\{ f^{\sharp\flat}, \leq^{\sharp\flat} \mid f \in L_{k+1} \right\}$ satisfies all $BASIC_k$ axioms. Moreover, $f^{\sharp\flat}$ are Σ^b_1-defined and $\leq^{\sharp\flat}$ is Δ^b_1-defined so we may freely apply $\Sigma^b_1 - PIND$ to formulas containing those symbols. But S^1_{k+1} readily proves that axioms from the list $BASIC_k$ uniquely determine symbols from the language L_{k+1}.■

7. Summary

We summarize results established in the previous sections in the following theorem:

Theorem 7.1 (Main) *For each $i, k \geq 1$ there exist interpretations $\flat : V^i_k(BD) \leadsto S^i_{k+1}$ and $\sharp : S^i_{k+1} \leadsto V^i_k(BD)$ such that:*

1. *for any* $\Sigma_j^{1,b}$*-formula* $A(x_1, \ldots, x_n)$ *in* $\mathcal{L}_k$ *(where* $j \geq 1$ *and all free first order variables in* A *are displayed),* $A^\flat(|x_1|, \ldots, |x_n|)$ *is equivalent in* S^1_{k+1} *to a* Σ_j^b*-formula,*

2. *for any* Σ_j^b*-formula* A *in* L_{k+1}*,* $A^\sharp$ *is equivalent in* $V_k^1(BD)$ *to a* $\Sigma_j^{1,b}$*-formula (again, provided* $j \geq 1$*),*

3. *for any formula* $A(x_1, \ldots, x_n)$ *in* $\mathcal{L}_k$ *(all free first order variables are displayed),*

$$V_k^1(BD) \vdash A^{\flat\sharp}(id(x_1), \ldots, id(x_n)) \equiv A(x_1, \ldots, x_n),$$

4. *for any formula* A *in* L_{k+1}*,*

$$S^1_{k+1} \vdash A^{\sharp\flat} \equiv A.$$

Proof. This is the content of Lemmas 4.2, 5.4, 6.1, 6.2.■

As an application we obtain the analog of Buss's main result for the theory $V_2^1(BD)$ as a *direct* consequence of Proposition 2.2:

Theorem 7.2 1. *For each function* $f(\vec{x}) \in EXPTIME$ *of polynomial growth rate there exist a* $\Sigma_1^{1,b}$*-formula* $A(\vec{x}, y)$ *and a first order term* $t(\vec{x})$*, both in* $\mathcal{L}_2$ *such that:*

   (a) $V_2^1(BD) \vdash \exists y \leq t(\vec{x})\ A(\vec{x}, y)$,

   (b) $V_2^1(BD) \vdash A(\vec{x}, y) \wedge A(\vec{x}, z) \ \supset\ y = z$,

   (c) *for all* $\vec{n}$*,* $\mathbf{N} \models_{BD} A(\vec{n}, f(\vec{n}))$ *where* $\models_{BD}$ *corresponds to the model where second order variables range over all* α *with finitely many ones.*

2. *Conversely, suppose* $V_2^1(BD) \vdash \forall\vec{x}\exists y\ A(\vec{x}, y)$ *where* $A(\vec{x}, y)$ *is a* $\Sigma_1^{1,b}$*-formula with all free variables displayed. Then there exist a* $\Delta_1^{1,b}$*-formula* $B(\vec{x}, y)$*, a first order term* $t(\vec{x})$ *(in* $\mathcal{L}_2$*) and an* $EXPTIME$ *function* $f(\vec{x})$ *of polynomial growth rate such that:*

   (a) $V_2^1(BD) \vdash B(\vec{x}, y) \supset A(\vec{x}, y)$,

   (b) $V_2^1(BD) \vdash \exists y \leq t(\vec{x})\ B(\vec{x}, y)$,

   (c) $V_2^1(BD) \vdash B(\vec{x}, y) \wedge B(\vec{x}, z) \ \supset\ y = z$,

(d) for all $\vec{n}$, $\mathbf{N} \models_{BD} B(\vec{n}, f(\vec{n}))$.

Proof. We start with the part 2.

By Theorem 7.1 1, $A^\flat(|\vec{x}|, |y|)$ is equivalent in S_3^1 to a Σ_1^b-formula. Apply Proposition 2.2 2 with $k = 3$ to the formula $A^\flat(|\vec{x}|, |y|)$ and find corresponding triple (B, t, f) which we redenote as $(B'(\vec{x}, y), t'(\vec{x}), f'(\vec{x}))$. We define (B, t, f) by

$$\begin{aligned} t(\vec{x}) &\rightleftharpoons (t')^-(\vec{x}), \\ B(\vec{x}, y) &\rightleftharpoons \exists\beta\left(B'^\sharp\left(exp(\vec{x}), \beta|_{t(\vec{x})}\right) \wedge \ell\left(\beta|_{t(\vec{x})}\right) = y\right), \\ f(\vec{x}) &\rightleftharpoons \left|f'\left(2^{\vec{x}}\right)\right|. \end{aligned}$$

Let us check that these B, t, f possess all required properties.

Since $f'(\vec{n})$ is computable in time $\exp\left(\log^{O(1)}|\vec{n}|\right)$, $f'\left(2^{\vec{n}}\right)$ and hence $f(\vec{n})$ are computable in time $\exp\left(|\vec{n}|^{O(1)}\right)$. Now $|f'(\vec{n})| \leq \exp\left(\log^{O(1)}|\vec{n}|\right)$ implies $f(\vec{n}) = \left|f'\left(2^{\vec{n}}\right)\right| \leq \exp\left(|\vec{n}|^{O(1)}\right)$ that is $|f(\vec{n})| \leq |\vec{n}|^{O(1)}$. Which means f has polynomial growth rate.

Proposition 2.2 2a says that $S_3^1 \vdash B'(\vec{x}, y) \supset A^\flat(|\vec{x}|, |y|)$. Applying $\sharp$ gives us

$$V_2^1(BD) \vdash B'^\sharp(\vec{\alpha}, \beta) \supset A^{\flat\sharp}(|\vec{\alpha}|, |\beta|).$$

Substituting here $exp(\vec{x})$ for $\vec{\alpha}$ and $\beta|_{t(\vec{x})}$ for β, we will have $V_2^1(BD) \vdash B(\vec{x}, y) \supset A^{\flat\sharp}(id(\vec{x}), id(y))$. Which, along with Theorem 7.1 3, gives us 2a.

Similarly, applying $\sharp$ to Proposition 2.2 2c, we have

$$V_2^1(BD) \vdash \exists\beta \leq t'^\sharp(\vec{\alpha})\ B'^\sharp(\vec{\alpha}, \beta).$$

If β is such that $\beta \leq t'^\sharp(exp(\vec{x}))$ and $B'^\sharp(exp(\vec{x}), \beta)$ then

$$\ell(\beta) \leq \ell\left(t'^\sharp(exp(\vec{x}))\right) \leq \text{ (by Lemma 5.2) } (t')^-(\vec{x}) = t(\vec{x})$$

hence $\beta|_{t(\vec{x})} = \beta$ and $B(\vec{x}, \ell(\beta))$. This proves 2b.

2c is proved similarly.

B is equivalent to a $\Sigma_1^{1,b}$-formula by Theorem 7.1 2. Already proven 2b and 2c imply its $\Pi_1^{1,b}$-representation $V_2^1(BD) \vdash B(\vec{x}, y) \equiv \forall z \leq t(\vec{x})\,(B(\vec{x}, z) \supset z = y)$. Hence B is in $\Delta_1^{1,b}$.

Lastly, $\mathbf{N} \models B'(\vec{n}, f'(\vec{n}))$ implies $\mathbf{N} \models_{BD} B'^\sharp(id(\vec{n}), id(f'(\vec{n})))$ since $\sharp$ is sound with respect to $\models_{BD}$. Substitute here $2^{\vec{n}}$ for $\vec{n}$ and let $\eta \rightleftharpoons id\left(f'\left(2^{\vec{n}}\right)\right)$. Then, as in the proof of 2b, $\mathbf{N} \models_{BD} \eta|_{t(\vec{n})} = \eta$ and hence $\mathbf{N} \models_{BD} B(\vec{n}, \ell(\eta))$. We only have to note now that $\ell(\eta) = \left|f'\left(2^{\vec{n}}\right)\right| = f(\vec{n})$.

The proof of part 2 is completed.

1 Let $f'(\vec{n}) \rightleftharpoons 2^{f(|\vec{n}| \dot{-} 1)} \dot{-} 1$. $f(|\vec{n}|)$ is computable in time

$$\exp\left(\log^{O(1)} |\vec{n}|\right)$$

and $f(|\vec{n}|) \leq \exp\left(\log^{O(1)} |\vec{n}|\right)$. Hence $f'(\vec{n})$ is computable in time

$$\exp\left(\log^{O(1)} |\vec{n}|\right) \leq |t'(\vec{n})|$$

for some term t' of the language L_3. Apply Proposition 2.2 1 with $k = 3$ to find corresponding A'. Now the same proof allows us to lift f', t', A' to some $\tilde{f}, t, A$ such that 1a, 1b and $\mathbf{N} \models_{BD} A(\vec{n}, \tilde{f}(\vec{n}))$. But $\tilde{f}(\vec{n}) = \left|f'\left(2^{\vec{n}}\right)\right| = f\left(\left|2^{\vec{n}}\right| \dot{-} 1\right) = f(\vec{n})$. The proof of part 1 is completed.■

Remark 3 The same argument can be applied to the case $i = 1$ i.e. $V_1^1(BD)$ vs. S_2^1. This will imply that $V_1^1(BD)$ can $\Sigma_1^{1,b}$-define exactly those functions $f(\vec{n})$ which are computable in time $2^{O(|\vec{n}|)}$ and have *linear* growth rate. Similarly, $V_1(BD)$ (which can be viewed as a weak monadic extension of $I\Delta_0$) $\Sigma^{1,b}$-defines exactly those functions which belong to the linear exponential time hierarchy $E \cup E^{NE} \cup E^{NE^{NE}} \cup \ldots$ and have linear growth rate (here $E \rightleftharpoons DTIME\left[2^{O(n)}\right]$) etc.

8. Conservativity results

In this section we will show that the theories $\widetilde{V}_k^i, \overset{\circ}{V}{}_k^i, V_k^i(bd)$ and $V_k^i(BD)$ prove the same $\Sigma^{1,b}$-formulae and the same closed $\forall^0\exists^0\Sigma^{1,b}$-formulae (the superscript "0" indicates that the quantifiers are first order). This implies that in the part concerning such formulae, the equivalence between $V_k^i(BD)$ and S_{k+1}^i established in previous sections is extended to $\widetilde{V}_k^i, \overset{\circ}{V}{}_k^i, V_k^i(bd)$.

Lemma 8.1 $\overset{\circ}{V}{}_k^i$ *is* $\Sigma^{1,b}$*-conservative over* $V_k^i(bd)$.

Proof. In the course of this proof it will be convenient for us to change the framework and consider $\overset{\circ}{V}{}^i_k$ and $V^i_k(bd)$ as Gentzen style second order theories rather than merely many sorted theories.

Assume $\overset{\circ}{V}{}^i_k \vdash A$ where A is bounded. Consider the theory $\overset{\circ}{V}{}^i_k(\delta)$ which is obtained from $\overset{\circ}{V}{}^i_k$ by adding to the language new predicate symbols δ for all $\Delta^{1,b}_1$ predicates and replacing the scheme $\Delta^{1,b}_1 - CA$ by $\Sigma^{1,b}_0(\delta) - CA$ (see [Bus86, §9.7]). By [Bus86, Corollary 9.21] there exists a Gentzen style $\overset{\circ}{V}{}^i_k(\delta)$-proof of the sequent $\rightarrow A$ such that all formulae in this proof are bounded. We can eliminate now in this proof symbols δ to get a proof *in* $\overset{\circ}{V}{}^i_k$ with the same property (compare [Bus86, Proposition 9.18]). So we only have to check that the comprehension rules

$$\frac{\Gamma \rightarrow \Delta, F(V)}{\Gamma \rightarrow \Delta, \exists\phi F(\phi)} \qquad\qquad \frac{F(V), \Gamma \rightarrow \Delta}{\forall\phi F(\phi), \Gamma \rightarrow \Delta}$$

where V is a $\Delta^{1,b}_1$-abstract *and F is bounded* are admissible in $V^i_k(bd)$.

To see this, note that the proofs of theorem 3.6 1 and (14) generalize to showing that for F, V as above there exists a first order term t such that $V^0_k(bd) \vdash F(V) \equiv F(V|_t)$ where $V|_t$ is defined in the obvious way. But the rules

$$\frac{\Gamma \rightarrow \Delta, F(V|_t)}{\Gamma \rightarrow \Delta, \exists\phi F(\phi)} \qquad\qquad \frac{F(V|_t), \Gamma \rightarrow \Delta}{\forall\phi F(\phi), \Gamma \rightarrow \Delta}$$

are equivalent to $\Delta^{1,b}_1 - BCA$ which is provable in $V^1_k(bd)$ by Theorem 3.5.∎

Define the translation $*$ from the set of formulae in the language $\mathcal{L}_k$ to itself by relativizing all second order quantifiers $\exists\alpha, \forall\alpha$ to the domain $\exists x \forall y\, (\alpha(y) \supset y < x)$.

Lemma 8.2 *For each bounded formula A, $V^0_k(bd) \vdash A^* \equiv A$.*

Proof. Induction on complexity of A. The only nontrivial case $A \equiv \exists\alpha\ B(\alpha)$ is taken care of by Theorem 3.6 1 and (14) like in the proof of previous lemma.∎

Corollary 8.3 *$*$ is an interpretation of $V^i_k(BD)$ in $V^i_k(bd)$.*

Proof. Obviously follows from Lemma 8.2.■

Corollary 8.4 *$V_k^i(BD)$ is $\Sigma^{1,b}$-conservative over $V_k^i(bd)$.*

Theorem 8.5 *$\widetilde{V}_k^i, \overset{\circ}{V}{}_k^i, V_k^i(bd)$ and $V_k^i(BD)$ prove the same bounded formulae.*

Proof. Immediate from Lemma 8.1 and Corollary 8.4.■

Theorem 8.6 *All four theories $\widetilde{V}_k^i, \overset{\circ}{V}{}_k^i, V_k^i(bd)$ and $V_k^i(BD)$ prove the same closed $\forall^0\exists^0\Sigma^{1,b}$-formulae.*

Proof. Let V_k^i be one of the two theories $\overset{\circ}{V}{}_k^i, V_k^i(BD)$ which proves $\forall\vec{x}\exists y_1, \ldots, y_r\ A(\vec{x}, \vec{y})$ where A is a bounded formula with all free variables displayed. Decoding the vector $\vec{y}$ by $\langle y_1, \ldots, y_r\rangle$, we may assume that $r = 1$. As before, it suffices to check that

$$V_k^i(bd) \vdash \exists y\ A(\vec{x}, y). \tag{34}$$

By the extension of Parikh's theorem [Par71] to second order Bounded Arithmetic in the case $V_k^i \equiv \overset{\circ}{V}{}_k^i$ and by Theorem 7.2 2 in the case $V_k^i \equiv V_k^i(BD)$, we have a first order term $t(\vec{x})$ such that $V_k^i \vdash \exists y \leq t(\vec{x})\ A(\vec{x}, y)$. Applying Theorem 8.5 gives us $V_k^i(bd) \vdash \exists y \leq t(\vec{x})\ A(\vec{x}, y)$ which implies (34).■

9. Concluding remarks

In this paper we considered bounded domain versions $V_k^i(BD)$ of Buss's theories V_k^i, developed a logical formalism allowing one to understand their power and showed that these versions talk of essentially the same domain and have exactly the same power as their first order counterparts S_{k+1}^i (Theorem 7.1). The bounded domain versions do not differ much from original Buss's (unbounded domain) theories while we are concerned with bounded (or "almost bounded") formulae (Theorems 8.5, 8.6). Also they are robust with respect to the choice of the bounded comprehension axiom scheme (Theorem 3.5). The analogue of the latter property is not expected in the case of unbounded domain.

Probably this equivalence can be without difficulties extended to higher order theories. A typical result might say that the "bounded domain" theory obtained from S_k^i by allowing functionals of type $\leq l$ is equivalent to S_{k+l}^i. The equivalence should interpret functionals of type $j \leq l$ by those integers x for which the j-th fold exponent exists (compare with [CT86]). However I did not try to develop this systematically.

10. Acknowledgment

I am grateful to Lev Beklemishev and anonymous referee for extensive comments made on an earlier version of this paper. I also wish to express my sincere gratitude to Samuel Buss and Gaisi Takeuti for pointing the sources [All89, Clo89, Tak90, Tak91] out to me.

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