Complexity Theory

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You may work together on solving homework problems, but please put all the names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. If you find any of the solutions on-line, it is OK but then you must completely understand the proof, explain it in your own words and include the URL.

The due date below pertains to e-mail submissions as a PDF file prepared from a TeX source: this is much preferred format. Handwritten work must be submitted by the beginning of class a day before the deadline.

Homework 1, due May 8

1. Let

$$\Phi_e(x) \stackrel{\text{def}}{=} \begin{cases} \text{the product of the number of } \mathsf{LEFT} \text{ instructions and the number of } \mathsf{RIGHT} \\ \text{instructions the Turing machine } M_e \text{ executes on the input } x \text{ if it halts,} \\ \text{undefined otherwise.} \end{cases}$$

   Prove that Φ_e is an abstract complexity measure, i.e., that it satisfies Blum axioms.

2. Determine the complexity of the problem $\mathsf{HORN}^+\mathsf{SAT}$: given a CNF ϕ in which every clause is either Horn or positive[1], decide if ϕ is satisfiable.

[1] does not contain negated literals

3. A *perfect matching* in a graph with $2n$ vertices is a set of n edges that do not have any vertices in common. PERFECT MATCHING is the language that consists of all graphs with even number of vertices that possess at least one perfect matching.

   Construct an *explicit*[2], *direct* and *combinatorial* many-one Karp reduction from PERFECT MATCHING to INDEPENDENT SET.

4. A *prime machine* is a non-deterministic machine that outputs YES if and only if the number of computational paths leading to the state q_{accept} is a prime number. **PrimeP** is the class of all languages recognizable by poly-time prime machines.

   Prove that there exist two oracles A and B such that $\mathbf{NP}^A = \mathbf{PrimeP}^A$ and[3] $\mathbf{NP}^B \neq \mathbf{PrimeP}^B$.

5. Which of the two classes **PSPACE** and **POLYLOGSPACE** $\stackrel{\text{def}}{=} \bigcup_{k>0} \mathbf{SPACE}((\log n)^k)$ has a log-space complete problem (prove your answers, of course)? What conclusions about relations to **P** and **NP** can you draw from this?

[2] any smart deviations, like a reference to Cook-Levin theorem or to the fact that PERFECT MATCHING is in **P** shall incur quite drastic reduction of credit

[3] You can choose the direction. If you do both, it will bring you a few points of extra credit.