Complexity Theory

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You may work together on solving homework problems, but please put all the names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. If you find any of the solutions on-line, it is OK but then you must completely understand the proof, explain it in your own words and include the URL.

The due date below pertains to e-mail submissions as a PDF file prepared from a TeX source: this is much preferred format. Handwritten work must be submitted by the beginning of class a day before the deadline.

Homework 2, due May 22

1. Recall that $\mathbf{R}$ consists of all languages L for which there exists a polynomial time non-deterministic Turing machine M with the following acceptance conditions:

$$\begin{cases} x \in L \Longrightarrow \mathbf{P}[M \text{ accepts } x] \geq 1/2 \\ x \notin L \Longrightarrow \mathbf{P}[M \text{ rejects } x] = 1. \end{cases}$$

   Prove that $\mathbf{R}^{\mathbf{BPP}} = \mathbf{BPP}^{\mathbf{R}}$.

2. An r-*graph* is defined similarly to an ordinary graph with the difference that its edges[1] are unordered r-tuples of vertices rather than just pairs.

   Prove that the problem 3-GRAPH ISOMORPHISM is many-one Karp reducible to GRAPH ISOMORPHISM.

[1] often called *hyper*edges

3. Let us identify $\{0,1\}^*$ with the integers $\mathbb{N}$ in any reasonable way, say (for concreteness) by the function appending 1 in front of the string and interpreting the result as a number written in binary. Let

$$\mathbf{REPEL} \stackrel{\text{def}}{=} \left\{ L \subseteq \mathbb{N} \,\middle|\, \forall x, y \in L \left(x < y \Longrightarrow y \geq x \left(1 + \frac{1}{\log_2 x} \right) \right) \right\}.$$

   Prove that $\mathbf{P}^{\mathbf{REPEL}} = \mathbf{P}/poly$.

4. Prove Spira's theorem for the monotone basis $\{\wedge, \vee\}$. That is, prove that for any monotone formula F of size s there exists an equivalent monotone formula of depth $O(\log s)$.

5. Let G be a *fixed* finite group whose elements are identified with binary strings via an arbitrary embedding $G \rightarrowtail \{0,1\}^k$. Let $G\Pi_n$ be a Boolean function in kn variables defined by $G\Pi_n(X_1, \ldots, X_n) = 1$ ($X_i \in \{0,1\}^k$) iff $X_i \in G$ ($1 \leq i \leq n$) and $X_1 X_2 \cdots X_n = 1$.

   Prove that[2] $BPP(G\Pi_n) \leq O(n)$.

[2]Recall that $BPP(f)$ is the minimal possible size of a branching program computing f.