Complexity Theory

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You may work together on solving homework problems, but please
put all the names clearly at the top of the assignment. Everyone
must turn in their own independently written solutions. If you
find any of the solutions on-line, it is OK but then you must completely
understand the proof, explain it in your own words and include the
URL.
   The due date below pertains to e-mail submissions as a PDF file
prepared from a TeX source: this is much preferred format. Handwritten
work must be submitted by the beginning of class a day before the
deadline.
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Homework 3, due June 5

1. Recall that for a basis B the formula size $L_B(f)$ is defined as the minimum number of computational gates in a formula over the basis B computing the function f. Let $f_1(X_1), \ldots, f_m(X_m)$ be arbitrary non-constant Boolean functions depending on pairwise disjoint sets of variables.

   (a) Prove that $L_{B_2}(f_1(X_1) \vee \ldots \vee f_m(X_m)) \geq \frac{1}{2}\sum_{i=1}^m L_{B_2}(f_i(X_i))$, where B_2 is the basis consisting of all binary functions.

   (b) Prove that $L_{\{\neg,\wedge,\vee\}}(f_1(X_1) \oplus \ldots \oplus f_m(X_m)) \geq \Omega(m^2)$.

2. Prove that $\mathrm{Corr}(\mathrm{PARITY}_n, \mathrm{MAJ}_n) = \Theta(n^{-1/2})$.

   `Note.` You need to prove *both* upper and lower bounds.

3. Prove that $||A|| \geq \sqrt{N}$ for any $\{\pm 1\}$ $N \times N$ matrix A.

4. For a function $f : X \times Y \longrightarrow \{0, 1\}$, $\chi_1(f)$ is the minimal number of **disjoint** 1-rectangles covering $f^{-1}(1)$.

   Prove that $C(f) \leq O(\log \chi_1(f)^2)$.

5. Prove that $U(\mathrm{DISJ}_n) \leq O(\log n)$, where $U(f_n)$ is the *unbounded* error communication complexity of a function $f_n : \{0, 1\}^n \times \{0, 1\}^n \longrightarrow \{0, 1\}$.