Honors Discrete Mathematics

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Prove all of your answers with reasonable degree of mathematical rigor. If you work with others put their names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged: if you do it nonetheless, you *must* cite your source and, as the very least, explain the solution in your own words.

Homework is due at the beginning of Wednesday class *unless* submitted by e-mail as a PDF file prepared from a TeX source. Electronic submissions conforming to these standards (no scans please!) are encouraged and accepted until Wednesday midnight by Leo at lenacore@uchicago.edu.

Homework 1, due October 17

1. Prove that there are constants $\epsilon > 0$ and $C > 0$ such that for all positive integers n we have the inequality

   $$\epsilon n^{2019} \leq 1^{2018} + \cdots + n^{2018} \leq C n^{2019}.$$

   Note. For full credit the constants $\epsilon > 0$ and $C > 0$ must be completely explicit but they need not necessarily be optimal.

2. We are given $n \geq 2$ different points on the plane. Prove that the number of lines defined[1] by these points is either 1 or $\geq n$.

3. An integer n is called *square-free* if it is not divisible by any integer of the form m^2 with $m > 1$.

[1] A line is *defined* if it contains at least two points from the given family.

(a) Is the set of all non-zero square-free integers partially ordered by the divisibility relation $|$?

(b) Let $\mathbb{PSF}$ be the set of all *positive* square-free integers. Prove that $(\mathbb{PSF},\ |)$ is a partial order that, moreover, is isomorphic to the partial order $(\mathcal{P}_{<\omega}(\mathbb{N}),\ \subseteq)$ consisting of *finite* subsets of positive integers w.r.t. the set inclusion.

4. Prove that $\omega \neq \omega \cdot 2$. In other words, show that there is no one-to-one mapping between the ground sets that respects linear order.

5. A function $f : \mathbb{N} \longrightarrow \mathbb{N}$ satisfies the following two properties:

$$\begin{aligned} f(n) &= n - 7 \quad \text{if} \quad n > 49 \\ f(n) &= f(f(n+8)) \quad \text{if} \quad n \leq 49. \end{aligned}$$

Compute $f(9)$.