Honors Discrete Mathematics

Instructor: Alexander Razborov, University of Chicago
razborov@math.uchicago.edu
Course Homepage: www.cs.uchicago.edu/~razborov/teaching/autumn18.html

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Prove all of your answers with reasonable degree of mathematical rigor. If you work with others put their names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged: if you do it nonetheless, you *must* cite your source and, as the very least, explain the solution in your own words.

Homework is due at the beginning of Wednesday class *unless* submitted by e-mail as a PDF file prepared from a TeX source. Electronic submissions conforming to these standards (no scans please!) are encouraged and accepted until Wednesday midnight by Leo at lenacore@uchicago.edu.

Homework 3, due October 31

Problems 1-3 were kindly contributed by Professor Kurtz.

1. Show that the relation "$f(x)$ is $\Theta(g(x))$" is an equivalence relation on functions $f : \mathbb{N} \longrightarrow \mathbb{R}_{>0}$.

2. Solve the recurrence

$$f_n = \begin{cases} -1, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ 8f_{n-1} - 15f_{n-2}, & \text{otherwise.} \end{cases}$$

3. Solve the recurrence

$$f_n = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n = 1 \\ 6, & \text{if } n = 2 \\ 4f_{n-1} - f_{n-2} - 6f_{n-3}, & \text{otherwise.} \end{cases}$$

4. Recall that *Minkowski sum and product* of two **sets** (not necessarily ideals) A, B in a commutative ring R are defined as

$$A + B \stackrel{\text{def}}{=} \{a + b \mid a \in A, b \in B\}$$
$$A \cdot B \stackrel{\text{def}}{=} \{a \cdot b \mid a \in A, b \in B\}.$$

   (a) Identify those axioms of a commutative ring that hold in the structure $\langle \mathcal{P}(R), +, \cdot \rangle$ ($\mathcal{P}(R)$ is the family of all subsets of R). Like in 2-4, you will have to figure out first what sets play the roles of 0 and 1.

   (b) An element $A \in \mathcal{P}(R)$ is *absorbing* w.r.t. addition [w.r.t. multiplication] if $A+B = A$ [$AB = A$, respectively] for any $B \in \mathcal{P}(R)$. Describe all absorbing elements with respect to addition and all absorbing elements with respects to multiplication.

   (c) The same notion can be defined in the ring R itself. The same question: describe all absorbing elements in R with respect to each operation.

5. Prove that $\phi(m)\phi(n) \leq \phi(mn)$ for any integers $m, n \geq 2$ (ϕ is Euler's function) and characterize those pairs (m, n) for which the equality holds.