Honors Discrete Mathematics

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Prove all of your answers with reasonable degree of mathematical rigor. If you work with others put their names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged: if you do it nonetheless, you *must* cite your source and, as the very least, explain the solution in your own words.

Homework is due at the beginning of Wednesday class *unless* submitted by e-mail as a PDF file prepared from a TeX source. Electronic submissions conforming to these standards (no scans please!) are encouraged and accepted until Wednesday midnight by Leo at lenacore@uchicago.edu.

Homework 1, due October 16

1. Prove that
$$1! + 2! + \ldots + n! \leq \frac{3}{2}n!,\ n \in \mathbb{N}.$$

2. A function $f : \mathbb{N} \longrightarrow \mathbb{N}$ satisfies the following two properties:
$$\begin{aligned} f(n) &= n - 6 \quad \text{if} \quad n > 36 \\ f(n) &= f(f(n+7)) \quad \text{if} \quad n \leq 36. \end{aligned}$$

   Compute $f(8)$.

3. Let S consist of all finite subsets $A \subseteq \mathbb{N}$, and let $A \preceq B$ if and only if for **any** non-negative function $f : \mathbb{N} \longrightarrow \mathbb{R}_{\geq 0}$ we have $\sum_{x \in A} f(x) \leq \sum_{x \in B} f(x)$. Prove that $(S, \preceq)$ is a poset and describe it explicitly.

4. Let $S \stackrel{\text{def}}{=} \{11, 12, 13, 21, 22, 23\}$ be partially ordered by the following relation: $\overline{x_1 x_2} \preceq \overline{y_1 y_2}$ if and only if $x_1 = y_1 \ \wedge \ x_2 \leq y_2$. How many different total extensions does it have?

5. Let $(S, \leq)$ be a well-ordered set, and let $f : S \longrightarrow S$ be injective $(\forall x, y \in S(f(x) = f(y) \longrightarrow x = y))$ and monotone non-decreasing $(\forall x, y \in S(x \leq y \longrightarrow f(x) \leq f(y)))$. Prove that $\forall x(f(x) \geq x)$.

   `For extra credit:` Is the converse true, i.e. does this property completely characterize well-order sets? I can not see an immediate answer to this question...