Honors Discrete Mathematics

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Course Homepage: www.cs.uchicago.edu/~razborov/teaching/autumn19.html

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   Prove all of your answers with reasonable degree of mathematical
rigor.  If you work with others put their names clearly at the top
of the assignment.  Everyone must turn in their own independently
written solutions.  Shopping for solutions on the Internet is strongly
discouraged:  if you do it nonetheless, you *must* cite your source
and, as the very least, explain the solution in your own words.
   Homework is due at the beginning of Wednesday class unless submitted
by e-mail as a PDF file prepared from a TeX source.  Electronic
submissions conforming to these standards (no scans please!)  are
encouraged and accepted until Wednesday midnight by Leo at lenacore@uchicago.edu.
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Homework 3, due October 30

1. Another important operation on ideals is their intersection.

   For which of the three rings $\mathbb{Z}$, $\mathbb{R}[X]$, $\mathbb{R}[X,Y]$ we always have the ideal identity

   $$(I + J) \cap K = (I \cap K) + (J \cap K)?$$

2. Consider the following binary relation $R(a,b)$ on $\mathbb{R}$: $R(a,b)$ if and only if $a^3 - b^3 = a - b$. Prove that R is an equivalence relation.

3. Compute explicitly the largest integer $d = d(n,m)$ for which $n^d | \phi(n^m)$ ($n \geq 2$, $m \geq 1$).

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   Note:  A combinatorial proof avoiding prime factorization is
   highly preferred.
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4. Solve the following system of congruences:

$$\begin{cases} x \equiv 1 \ (\mathrm{mod}\,2) \\ x \equiv 1 \ (\mathrm{mod}\,3) \\ x \equiv 1 \ (\mathrm{mod}\,5) \\ x \equiv 2 \ (\mathrm{mod}\,7) \\ x \equiv 2 \ (\mathrm{mod}\,11) \\ x \equiv 2 \ (\mathrm{mod}\,13). \end{cases}$$

5. Let us call a function $f : X \longrightarrow Y$ *gorgeous* if for any function $g : Z \longrightarrow Y$ with $|Z| \leq 19$ there exists an injective function $h : Z \rightarrowtail X$ such that $g = f \circ h$. Let us call f *co-gorgeous* if for any $g : X \longrightarrow Z$, again with $|Z| \leq 19$, there exists a surjective $h : Y \twoheadrightarrow Z$ such that $g = h \circ f$.

   (a) Describe the set of those positive integers n for which there exists a gorgeous function $f : [100] \longrightarrow [n]$.

   (b) Describe the set of those positive integers n for which there exists a co-gorgeous function $f : [100] \longrightarrow [n]$.