Honors Discrete Mathematics

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Course Homepage: www.cs.uchicago.edu/~razborov/teaching/autumn19.html

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   Prove all of your answers with reasonable degree of mathematical
rigor.  If you work with others put their names clearly at the top
of the assignment.  Everyone must turn in their own independently
written solutions.  Shopping for solutions on the Internet is strongly
discouraged:  if you do it nonetheless, you *must* cite your source
and, as the very least, explain the solution in your own words.
   Homework is due at the beginning of Wednesday class unless submitted
by e-mail as a PDF file prepared from a TeX source.  Electronic
submissions conforming to these standards (no scans please!)  are
encouraged and accepted until Wednesday midnight by Leo at lenacore@uchicago.edu.
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Homework 6, due November 20

1. Let $n \geq 3$ be an integer. Construct n events $E_1, \ldots, E_n$ in the same sample space such that $p(E_{i_1} \wedge \ldots \wedge E_{i_k}) = p(E_{i_1}) \cdot \ldots \cdot p(E_{i_k})$ for any $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ with $k < n$ but $E_1, \ldots, E_n$ are not mutually independent.

2. Any cowboy in town is either good, that happens with probability 1/2, or bad with probability 1/6 or ugly with probability 1/3. Good cowboys shoot at strangers with probability 20%, ugly – with probability 30%, and bad cowboys shoot with probability 90%. You just met one on a street, and he shot you. What is the probability that you were shot by a good one?

3. Let X and Y be two independent variables uniformly distributed in the set $\{1, 2, 4, \ldots, 2^n\}$. Compute $E\left(\frac{\log_2 X}{Y}\right)$ as a closed-form expression.

4. Let X and Y be two independent variables in the same sample space.

   (a) Prove that $V(XY) \geq V(X)V(Y)$.

   (b) Prove that this bound is tight if and only if at least one of the following three holds:

      i. one of the variables is identically zero.

      ii. both variables are constant.

      iii. both variables have zero expectation.

   (c) Give an example showing that the bound may be not true without the independence assumption.

5. Let $X \stackrel{\text{def}}{=} |\text{im}(f)|$, where $f : [m] \longrightarrow [n]$ is picked uniformly at random among all functions from $[m]$ to $[n]$, and let $c \stackrel{\text{def}}{=} E(X)$. Prove that

$$p(X \geq 2c) \leq 1/c - 1/n.$$