Honors Discrete Mathematics

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Course Homepage: www.cs.uchicago.edu/~razborov/teaching/autumn19.html

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   Prove all of your answers with reasonable degree of mathematical
rigor.  If you work with others put their names clearly at the top
of the assignment.  Everyone must turn in their own independently
written solutions.  Shopping for solutions on the Internet is strongly
discouraged:  if you do it nonetheless, you *must* cite your source
and, as the very least, explain the solution in your own words.
   Homework is due at the beginning of Wednesday class unless submitted
by e-mail as a PDF file prepared from a TeX source.  Electronic
submissions conforming to these standards (no scans please!)  are
encouraged and accepted until Wednesday midnight by Leo at lenacore@uchicago.edu.
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Homework 7, due December 4

1. Assume that $d_1, \dots, d_n$ is the degree sequence of a simple graph. Prove that the following sequences are also degree sequences of a simple graph (order does not matter):

   (a) $d_1+2, d_1+2, d_2+2, d_2+2, d_3, d_3, \dots, d_n, d_n$;

   (b) $k, d_1+1, d_2+1, \dots, d_k+1, d_{k+1}, \dots, d_n$;

   (c) $n-1-d_1, n-1-d_2, \dots, n-1-d_n$.

2. Give a closed-form expression for the number of $(u, v)-$paths of length $n+2$ in the cycle graph $C_{2n},\ n \geq 2$, where u and v are two opposite vertices.

3. Give an example of two simple graphs that:

   (a) have the same degree sequences;

(b) for any given $r \geq 2$ have the same number of copies of K_r;

(c) for any given $\ell \geq 3$ have the same number of induced copies of C_ℓ

but nonetheless are not isomorphic to each other.

4. Compute $\mathrm{ex}(5; K_4^{(3)})$. In words: how many edges can a 3-graph on 5 vertices have if it is known that it does not contain any complete subgraph on 4 vertices?

5. Let G be a graph on 5 vertices. Prove that if $\omega(G) = \alpha(G)$ then $\chi(G) = 3$.

6. Let G be the complement of a tree on n vertices, $n \geq 2$. Describe all possible values $\chi(G)$ may take.

7. The *hypercube graph* Q_n has $V(Q_n) \stackrel{\text{def}}{=} \{0,1\}^n$, two strings x and y being adjacent if they differ in exactly one coordinate.

   Prove that if $n > 1$ then Q_n has a Hamiltonian circuit.