Honors Discrete Mathematics

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Autumn Quarter, 2021

Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged. But if you accidentally stumble across it, then it will be accepted as long as you cite the source and explain the solution in your own words.

Homework 1, due October 13

1. Prove that
$$1^1 + 2^2 + 3^3 + \cdots + n^n \leq \frac{5}{4}n^n,\ n \in \mathbb{N}.$$

2. A function $f : \mathbb{N} \longrightarrow \mathbb{N}$ satisfies the following two properties:
$$\begin{aligned} f(n) = n - 5 \quad &\text{if} \quad n > 25 \\ f(n) = f(f(n+6)) \quad &\text{if} \quad n \leq 25. \end{aligned}$$
Compute $f(7)$.

3. The *product order* of two posets $(S, \leq_S)$ and $(T, \leq_T)$ is the poset $(S \times T, \leq)$ such that $(s,t) \leq (s',t')$ if and only if $s \leq_S s'$ and $t \leq_T t'$ (note the difference from the lexicographic product!).

   Consider the set $\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ partially ordered by the divisibility relation. Prove that it can be represented as a product order of two *linear* orderings.

4. The ordering $\leq^{\perp}$ *dual* to a (partial or linear) ordering $(S, \leq)$ is defined by reversing the sign: $s \leq^{\perp} t$ iff $t \leq s$.

   Prove that $(\mathbb{Q}_{>0}, \leq)$ is isomorphic to its dual.

5. How many different ordinals can be represented in the form $\alpha + \beta + \gamma$, where $\{\alpha, \beta, \gamma\}$ is a permutation of $\{1, \omega, \omega^2\}$?