Honors Discrete Mathematics

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Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged. But if you accidentally stumble across it, then it will be accepted as long as you cite the source and explain the solution in your own words.

Homework 2, due October 20

1. Describe **explicitly** all those linearly ordered sets S for which:
   (a) The lexicographic product $S \times_{\text{lex}} \mathbb{Z}$ is isomorphic to $\mathbb{Z}$;
   (b) The lexicographic product $\mathbb{Z} \times_{\text{lex}} S$ is isomorphic to $\mathbb{Z}$.

2. The *least common multiple* $\text{lcm}(a, b)$ is defined as the smallest positive integer that is divisible by both a and b.

   Prove the identity

$$\text{lcm}(\text{gcd}(a, b), c) = \text{gcd}(\text{lcm}(a, c), \text{lcm}(b, c)).$$

3. Consider the equation

$$\frac{1}{a} + \frac{1}{b} = \frac{5}{2021}.$$

   (a) Solve it in positive integers a, b.
   (b) Prove that there are no other solutions (up to a permutation of its components).

4. Prove that for any positive integer n, $\gcd(n^2+n+1, 3n+1) \in \{1, 7\}$.

5. For two ideals I and J in a commutative ring R, let

$$
\begin{aligned}
I + J &\stackrel{\text{def}}{=} \{a + b \mid a \in I,\ b \in J\} \\
I \cdot J &\stackrel{\text{def}}{=} \{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \mid n \in \mathbb{N},\ a_1, \ldots, a_n \in I,\ b_1, \ldots, b_n \in J\}\,.
\end{aligned}
$$

Prove that $I+J$ and $I \cdot J$ are also ideals and that $I \cdot (J+K) = I \cdot J + I \cdot K$ for any three ideals I, J, K.