Honors Discrete Mathematics

Instructor: Alexander Razborov, University of Chicago
razborov@uchicago.edu
Course Homepage: www.cs.uchicago.edu/~razborov/teaching/autumn21.html

Autumn Quarter, 2021

```
Prove all of your answers with reasonable degree of mathematical
rigor (feel free to ask us when in doubt).  If you work with others
put their names clearly at the top of the assignment, everyone must
turn in their own independently written solutions.  Shopping for
solutions on the Internet is strongly discouraged.  But if you accidentally
stumble across it, then it will be accepted as long as you cite
the source and explain the solution in your own words.
```

Homework 7, due December 3

1. Prove that for an integer $n \geq 1$ we have

$$\frac{n(n+1)}{2} < \sqrt{1 \cdot 2} + \sqrt{2 \cdot 3} + \ldots + \sqrt{n(n+1)} < \frac{n(n+2)}{2}.$$

2. For an integer $n \geq 1$, define a random set $\boldsymbol{B} \subseteq [2n+1]$ via the following experiment. First pick $\boldsymbol{A} \subseteq [2n+1]$ uniformly at random. Then let

$$\boldsymbol{B} \stackrel{\text{def}}{=} \begin{cases} \boldsymbol{A} & \text{if } |\boldsymbol{A}| \geq n+1 \\ [2n+1] \setminus \boldsymbol{A} & \text{if } |\boldsymbol{A}| \leq n \end{cases}$$

(in other words, $\boldsymbol{B}$ is the largest of the two sets: $\boldsymbol{A}$ and its complement). Compute $p(1 \in \boldsymbol{B})$ as a closed-form expression.

3. The *mean deviation* $MD(X)$ of a random variable X is defined as $E(|X - c|)$, where $c = E(X)$ is the expectation of X.

   (a) Prove that for any two random variables X and Y on the same sample space, $MD(X+Y) \leq MD(X) + MD(Y)$.

(b) Prove that if X and Y are additionally known to be independent, then this inequality is *always* strict, unless one of the variables X, Y is trivial (that is, takes one fixed value with probability 1).

4. Let $X \stackrel{\text{def}}{=} |\text{im}(f)|$, where $f : [m] \longrightarrow [n]$ is picked uniformly at random among all functions from $[m]$ to $[n]$, and let $c \stackrel{\text{def}}{=} E(X)$. Prove that

$$p(X \geq 2c) \leq 1/c - 1/n.$$

5. Assume that $d_1, \dots, d_n$ is the degree sequence of a simple graph. Prove that the following sequences are also degree sequences of a simple graph (order does not matter):

   (a) $n-1-d_1, n-1-d_2, \dots, n-1-d_n$;

   (b) $d_1+1, d_1+1, d_2+1, d_2+1, \dots, d_{k-1}+1, d_{k-1}+1, d_k, d_k, \dots, d_n, d_n$;

   (c) $2d_1, d_2, d_2, \dots, d_n, d_n$.

   `Note. For this problem I do` **`not`** `recommend to look up Erdös-Gallai theorem I mentioned in class (although it's a cool thing for its own sake).`

6. Let $a, b \in \mathbb{Z}_n$ be such that $2a \neq 0,\ 2b \neq 0$. Consider the graph $G_{n,a,b}$ on $\mathbb{Z}_n$ with the set of edges

$$\{(x, x+a) \mid x \in \mathbb{Z}_n\} \cup \{(x, x+b) \mid x \in \mathbb{Z}_n\}.$$

   When is this graph connected?

7. Recall that P_n is the path graph on n vertices (and hence of length $n-1$). Give a closed-form expression for the number of (u, v)-paths of length $n+1$ in it, where u and v are the end points of P_n.