Honors Discrete Mathematics

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`Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged, and using AI or StackExchange` **shall be considered academic dishonesty and treated appropriately.**

Homework 2, due October 23

1. Let us call a pair $(x, y) \in \mathbb{Z}^2$ *Bezoutish* if there exist non-zero integers a, b such that $ax + by = \gcd(a, b)$.

   Give an intrinsic description of the set of Bezoutish pairs.

2. Find all solutions to

$$2n^2 + 1 | n^3 + 1,\ n \in \mathbb{Z}.$$

   (**and** prove that there are no others).

3. Consider the structure $(\mathbb{N}, \oplus, \cdot)$, i.e. positive integers in which normal addition is replaced with the operation $x \oplus y \stackrel{\text{def}}{=} \gcd(x, y)$. What axioms of a commutative ring does this structure satisfy?

4. For two ideals I, J in a commutative ring R, let

$$\begin{aligned} I + J &\stackrel{\text{def}}{=} \{a + b \,|\, a \in I,\ b \in J\} \\ I \cdot J &\stackrel{\text{def}}{=} \{a_1b_1 + a_2b_2 + \cdots + a_nb_n \,|\, n \in \mathbb{N},\ a_1, \ldots, a_n \in I,\ b_1, \ldots, b_n \in J\}. \end{aligned}$$

(a) Prove that $I + J$ and $I \cdot J$ are also ideals (possibly trivial, that is equal to $\{0\}$ or R);

(b) prove that if $I + J = R$ then $I \cdot J = I \cap J$;

(c) prove the converse if $R = \mathbb{Z}$ and both ideals I, J are non-zero.

5. Let S be a finite set and $ER(S)$ be the family of all equivalence relations on S. Turn $ER(S)$ into a partial order $(ER(S), \preceq)$ by letting $\approx_1 \preceq \approx_2$ iff $\approx_1$ is *coarser* than $\approx_2$ meaning $\forall a, b\ (a \approx_2 b \Longrightarrow a \approx_1 b)$.

   Prove that if $|S| \geq 4$ then $(ER(S), \preceq)$ is not isomorphic to its dual.