Honors Discrete Mathematics

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Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged, and using AI or StackExchange **shall be considered academic dishonesty and treated appropriately.**

Homework 3, due October 30

1. Solve the following system of congruences:

$$\begin{cases} x \equiv 1 \pmod{2} \\ x \equiv 1 \pmod{3} \\ x \equiv 1 \pmod{5} \\ x \equiv 2 \pmod{7} \\ x \equiv 2 \pmod{11} \\ x \equiv 2 \pmod{13}. \end{cases}$$

2. Find a two-digit integer n such that the following is true. For any $a, b \in \mathbb{Z}_{1001}^*$, $a^n \equiv b^n \mod 1001$.

3. The ℓ_∞-*norm* of a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is defined as $\ell_\infty(\mathbf{x}) \stackrel{\text{def}}{=} \max_{1 \leq i \leq n} |x_i|$.

   How many ordered triples $\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle \in (\{-1, 1\}^n)^3$ of n-dimensional vectors with entries 1 or -1 are there with the property $\ell_\infty(\mathbf{x}+\mathbf{y}+\mathbf{z}) \leq 1$?

4. Let X, Y be non-empty finite sets. Call a function $f : X \longrightarrow Y$ *beautiful* if for any function $g : Z \longrightarrow Y$ with $|Z| \leq 2024$ there exists an injective function $h : Z \rightarrowtail X$ such that $g = f \circ h$. Call f *co-beautiful* if for any $g : X \longrightarrow Z$ with $|Z| \leq 2024$, there exists a surjective $h : Y \twoheadrightarrow Z$ such that $g = h \circ f$.

   Give an *intrinsic* description of beautiful and co-beautiful functions.

5. Let $X = Y = \mathbb{N}$, $f : X \rightarrowtail Y$ be given by $f : x \mapsto 2x$ and $g : Y \rightarrowtail X$ be given by $g : y \mapsto 3y$. Give an explicit description of the bijective mapping $X \rightarrowtail\!\!\!\!\!\twoheadrightarrow Y$ resulting from the proof of Cantor-Schröder-Bernstein theorem. For concreteness, assume that we apply the function f to bi-infinite chains.