Honors Discrete Mathematics

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Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged, and using AI or StackExchange **shall be considered academic dishonesty and treated appropriately.**

Homework 5, due November 13

1. Let $n \geq 1$ and $N_n \stackrel{\text{def}}{=} \frac{(n^2)!}{(n!)^{n+1}}$.

   (a) Prove that N_n is an integer.

   (b) Prove that $N_n \leq S(n^2, n)$.

2. Seventeen distinct points are chosen inside the 4-dimensional unit cube $[0,1]^4$. Prove that there are two points x and y among them within the Euclidean distance $\left(\rho(x,y) \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^4 (x_i - y_i)^2}\right)$ at most one of each other.

3. We are given $2n$ events $E_1, \ldots, E_{2n}$. Assume that $p(E_i) = p$ for all $i \in [2n]$, and for $i \neq j$ we have

$$p(E_i \wedge E_j) = \begin{cases} q, & \text{if } |i-j| \geq n+1; \\ 0 & \text{otherwise.} \end{cases}$$

   Compute $p(E_1 \vee \ldots \vee E_{2n})$, as a close form expression.

4. Let $\boldsymbol{A}, \boldsymbol{B} \subseteq [n]$ be two random subsets picked independently according to the uniform distribution. Compute $p(\boldsymbol{A} \cup \boldsymbol{B} = [n])$, as a close form expression.

5. For any $n \geq 2$ give an example of n events $E_1, \ldots, E_n$ that are not mutually independent but such that for any $i \in [n]$, $(n-1)$ events $E_1, \ldots, E_{i-1}, E_{i+1}, \ldots, E_n$ already are mutually independent.