Honors Discrete Mathematics

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`Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged, and using AI or StackExchange` **shall be considered academic dishonesty and treated appropriately.**

Homework 6, due November 20

1. You and $2n$ other people are forming two queues. Everyone (including you) chooses each of the queues with probability $1/2$, independently of each other. Compute, as a closed form expression, the probability that you are a part of the odd queue, that is the queue with an odd number of people.

2. A fair (cubical) die is rolled n times. Give a closed form expression for the probability that

   (a) the sum

   (b) the product

   of outcomes is divisible by 6.

3. Prove or disprove the following.

   Let X, Y be non-negative real-valued random variables and F be an event, all in the same sample space. Assume that $E(X|F) \geq E(X)$ and $E(Y|F) \geq E(Y)$. Then $E(XY|F) \geq E(XY)$.

4. Alice plays a slot machine until she wins k times (not necessarily in a row!) Compute the expectation of the number of plays as a closed form expression if the probability of winning in every round is p, where $0 < p < 1$.

5. Let $m \leq n$. Pick uniformly at random an *injective* function $f : [m] \rightarrowtail [n]$. Let $X \stackrel{\text{def}}{=} |\{ i \in [m] \mid f(i) = i \}|$ be the random variable that counts the number of fixed points in f. Compute $E(X)$ and $V(X)$ as closed form expressions.