Honors Discrete Mathematics

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`Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged, and using AI or StackExchange` **shall be considered academic dishonesty and treated appropriately.**

Homework 7, due December 5

1. Assume that $d_1, \ldots, d_n$ is the degree sequence of a simple graph. Prove that the following sequences are also degree sequences of a simple graph (order does not matter):

   (a) $n-1-d_1, n-1-d_2, \ldots, n-1-d_n$;

   (b) $d_1+1, d_1+1, d_2+1, d_2+1, \ldots, d_{k-1}+1, d_{k-1}+1, d_k, d_k, \ldots, d_n, d_n$;

   (c) $2d_1, d_2, d_2, d_3, d_3, \ldots, d_n, d_n$.

2. The *line graph* $L(G)$ of a simple graph $G = (V, E)$ has E as the vertex set, $e_1, e_2 \in E$ being connected if and only if they share a vertex in G.

   Which of the following statements are always true:

   (a) If G is regular then $L(G)$ is regular;

   (b) If $L(G)$ is regular then G is regular?

3. The *cartesian product* of two graphs $G = (V, E)$ and $H = (W, F)$ is defined as the graph $G \Box H$ with the vertex set $V \times W$ and the set of edges

$$\{ (\langle v_1, w \rangle, \langle v_2, w \rangle) \mid (v_1, v_2) \in E,\ w \in W \} \cup \{ (\langle v, w_1 \rangle, \langle v, w_2 \rangle) \mid v \in V,\ (w_1, w_2) \in F \}.$$

The *tensor product* $G \times H$ has the same vertex set but the set of edges is

$$\{ (\langle v_1, w_1 \rangle, \langle v_2, w_2 \rangle) \mid (v_1, v_2) \in E,\ (w_1, w_2) \in F \}.$$

Finally, the *lexicographic product* $G[H]$ still has the vertex set $V \times W$ and the set of edges is

$$\{ (\langle v_1, w_1 \rangle, \langle v_2, w_2 \rangle) \mid (v_1, v_2) \in E;\ w_1, w_2 \in W \} \cup \{ (\langle v, w_1 \rangle, \langle v, w_2 \rangle) \mid v \in V,\ (w_1, w_2) \in F \}.$$

Which of the following bounds on the diameter are true for arbitrary connected non-trivial (that is, on ≥ 2 vertices) graphs G and H:

$$\begin{aligned} \operatorname{diam}(G \Box H) &\leq \operatorname{diam}(G) + \operatorname{diam}(H) & (1) \\ \operatorname{diam}(G \times H) &\leq \max(\operatorname{diam}(G), \operatorname{diam}(H)) & (2) \\ \operatorname{diam}(G[H]) &\leq \operatorname{diam}(G)? & (3) \end{aligned}$$

4. Compute $\operatorname{ex}(5; K_4^{(3)})$. In words: how many edges can a 3-graph on 5 vertices have if it is known that it does not contain any complete subgraph on 4 vertices?

5. What is the smallest k for which the set of edges $\binom{[n]}{2}$ of the complete graph K_n can be partitioned into k subsets $E_1, \ldots, E_k$ in such a way that all k spanning subgraphs $([n], E_i)$ are bipartite?