Honors Discrete Mathematics

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Autumn Quarter, 2025

`Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged, and using AI or StackExchange is strictly prohibited, it` **shall be considered academic dishonesty and treated accordingly.**

Homework 1, due October 15

1. Let $A \subseteq \mathbb{N}$ be a finite set with n elements and let[1]

   $$A \cdot A \stackrel{\text{def}}{=} \{a_1 a_2 \mid a_1, a_2 \in A\}.$$

   Prove that

   $$|A \cdot A| \geq 2n - 1,$$

   and for every $n \geq 1$ give an example where this inequality is tight.

2. Prove that

   $$1^3 - 2^3 + 3^3 - 4^3 + \ldots + (2n+1)^3 \leq O(n^3).$$

3. A linearly ordered set $(S, \leq)$ is *self-dual* if there exists a one-to-one correspondence (i.e. bijection) $f : S \longrightarrow S$ that **reverses** the order, i.e. $f(s) \leq f(t)$ iff $t \leq s$.

   Which of the folowing sets[2] are self-dual and why: $\mathbb{Q}$, $\mathbb{Q}_{>0}$, $\mathbb{Q}_{\geq 0}$, $[0, 1]$?

[1] In this definition, a_1 and a_2 need not necessarily be distinct.

[2] The ordering is assumed to be standard in all examples.

4. Consider the partial ordering $([5], \preceq)$, where $x \preceq y$ iff $x = y$ or $y \geq x + 2$. How many different total extensions does it have?

5. (a) Prove that $(\omega + 1) \cdot n = \omega \cdot n + 1$ for any positive integer n.
   (b) Is it true that $(\omega + 1) \cdot \omega = \omega^2 + 1$?