Honors Discrete Mathematics

Instructor: Alexander Razborov, University of Chicago
razborov@uchicago.edu
Course Homepage: www.cs.uchicago.edu/~razborov/teaching/autumn25.html

Autumn Quarter, 2025

Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged, and using AI or StackExchange is strictly prohibited, it **shall be considered academic dishonesty and treated accordingly.**

Homework 2, due October 22

1. Let $x_1, \ldots, x_n \in \mathbb{N}$. Prove the inequality

$$\prod_{1 \leq i < j \leq n} \gcd(x_i, x_j) \leq \left(\prod_{i=1}^{n} x_i \right)^{\frac{n-1}{2}}.$$

2. Let $a \neq b \in \mathbb{N}$ be fixed. Prove that the set

$$\{\gcd(a+x, b+x) \mid x \in \mathbb{N}\}$$

is finite and determine its smallest and largest elements.

3. How many ways are there to endow the Abelian group $\mathbb{Z}$ with the structure of a commutative ring? In other words, how many ways are there to pick up a "unit" element $u \in \mathbb{Z}$ and a binary operation $* : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ in such a way that the structure $< \mathbb{Z}, 0, u, +, * >$ satisfies all axioms of a commutative ring?

4. Let R be a commutative ring, and $\mathcal{I}$ be the set of all its ideals (including trivial ones). For $I, J \in \mathcal{I}$, let

$$
\begin{aligned}
I + J &\stackrel{\text{def}}{=} \{a + b \mid a \in I,\ b \in J\} \\
I \cdot J &\stackrel{\text{def}}{=} \{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \mid n \in \mathbb{N},\ a_1, \ldots, a_n \in I,\ b_1, \ldots, b_n \in J\}.
\end{aligned}
$$

Prove that $I + J$ and $I \cdot J$ are also ideals and that these two operations on $\mathcal{I}$ satisfy[1] all axioms of a commutative ring except for one.

5. Define the binary relation $\approx$ on $\mathbb{Z}$ by letting $a \approx b$ iff $(a^3 - b^3) \mod 2025 = (a^4 - b^4) \mod 2025$. Prove that $\approx$ is an equivalence relation.

[1]you will have to figure out first what ideals play roles of 0 and 1