Honors Discrete Mathematics

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Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged, and using AI or StackExchange is strictly prohibited, it **shall be considered academic dishonesty and treated accordingly.**

Homework 5 (after "Arabian Nights"), due November 12

1. Prove that

$$\left(\left.1001^{1001^{\cdot^{\cdot^{\cdot^{1001}}}}}\right\}1001\text{ times}\right) = \left(\left.1001^{\cdot^{\cdot^{\cdot^{1001}}}}\right\}10\text{ times}\right) \bmod 720.$$

2. Recall that $p_m(n)$ is the number of **unordered** partitions of the integer n into at most m integers. Define $\bar{p}_m(n)$ similarly, with the following two differences:

   (a) The partition must have **precisely** m non-zero components,

   (b) These m non-zero components must be pairwise distinct.

   Prove that $\bar{p}_{10}(1001) = p_{10}(946)$.

3. (a) Let S be a set of integers ≥ 2 with $|S| = 1001$. Prove that S either contains an integer that factors into at least 11 primes (not necessarily distinct) or it has a subset T of size 101 in which no element divides any other element from T.

(b) Give an example showing that this conclusion does not necessarily hold when $|S| = 1000$.

4. A fair die is rolled 1001 times. Compute, as a closed form expression, the probability that the **product** of all outcomes is equal to 2 mod 6.

5. Give an example of events E_i ($i \in \mathbb{Z}_{1001}$) in the same sample space of your choice such that for any $i \in \mathbb{Z}_{1001}$ the events $E_i, E_{i+1}, \ldots E_{i+k}$ are mutually independent while E_i and E_{i+k+1} are not independent when:

   (a) $k = 12$

   (b) $k = 11$.