

CMSC 27130
Honors Discrete Mathematics

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Notation and Conventions

For this document, I adhere to the following conventions:

- The “natural numbers” are represented by \mathbb{N} and are equal to $\{1, 2, 3, \dots\}$.
- $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$.
- The notation $[n]$ represents the set $\{1, 2, \dots, n\}$ for any $n \in \mathbb{N}$.
- A *definition* defines a new term.
- A *theorem* is a main result.
- A *lemma* is a result used to build up to a theorem.
- A *corollary* is an immediate result of a theorem or lemma.
- A *remark* is a side comment or claim (usually unproven) that is interesting but usually not critical for the class.
- An *example* is an example that demonstrates the application of a previous definition of proven result.
- A *question* is a motivating question for the definitions and results that follow it.

Part I

Number Theory

1 The Euclidean Algorithm

Definition 1.1. An integer a is said to *divide* an integer b if $an = b$ for some $n \in \mathbb{Z}$. This is denoted $a \mid b$.

Definition 1.2. The *greatest common divisor* of two integers $a, b \in \mathbb{Z}$ is the largest positive integer d such that $d \mid a$ and $d \mid b$. The greatest common divisor of a and b is denoted $\gcd(a, b)$.

Example 1.3. The greatest common divisor of 6 and 8 is 2, and the greatest common divisor of 3 and 12 is 3. Thus, $\gcd(6, 8) = 2$ and $\gcd(3, 12) = 3$.

Theorem 1.4. If a and b are integers with $\gcd(a, b) > 1$, then the equation $ax + by = 1$ does not have any integer solutions (a, b) .

Definition 1.5. Integers a and b are said to be *relatively prime* if $\gcd(a, b) = 1$.

Theorem 1.6 (Bézout's theorem). If integers a and b are relatively prime, then there exists an integer solution (a, b) to the equation $ax + by = 1$.

Remark 1.7. Unlike **Theorem 1.4**, this direction is highly non-trivial. We will gradually build up the machinery necessary for its proof.

2 Mathematical Induction

Definition 2.1. Let P be a proposition defined for all natural numbers. The *principle of mathematical induction* states that if $P(1)$ is true and $P(k) \implies P(k + 1)$ for all $k \in \mathbb{N}$, then $P(n)$ is true for all $n \in \mathbb{N}$.

Remark 2.2. In inductive proofs, the step of proving $P(1)$ is usually called the *base case*. Assuming that $P(k)$ holds for an arbitrary $k \in \mathbb{N}$ is usually called the *inductive hypothesis* or *induction hypothesis*, and the step of proving $P(k) \implies P(k + 1)$ is usually called the *inductive step*.

Remark 2.3. Often times, we wish to prove a proposition for some integer $n \geq n_0$, where n_0 is not necessarily 1. It is valid to start the base case at an integer $n_0 \neq 1$, but the proposition will hold only for $n \geq n_0$. For example, we may prove $P(2)$ is true and that $P(k) \implies P(k+1)$ for $k \geq 2$, but, having done so, we will have proven only that $P(n)$ is true for $n \geq 2$, not all $n \in \mathbb{N}$ (namely, not $n = 1$). Finally, note that n_0 may be negative or zero as well.

Example 2.4. For all $n \in \mathbb{N}$, we have $1 + 2 + \dots + n = \frac{n(n+1)}{2}$.

Proof. We will prove this statement by induction. Our proposition P that we wish to prove is $P(n) := 1 + 2 + \dots + n = \frac{n(n+1)}{2}$. For the base case, $P(1)$, we have $\frac{1 \cdot (1+1)}{2} = \frac{2}{2} = 1$, so $P(1)$ holds. Now assume that $P(k)$ holds for some $k \in \mathbb{N}$. Then

$$\begin{aligned} 1 + 2 + \dots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) && \text{(inductive hypothesis)} \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2}, \end{aligned}$$

so $P(k) \implies P(k+1)$. Therefore, by the principle of mathematical induction, $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ holds for all $n \in \mathbb{N}$. \square

Definition 2.5. A collection of sets A_1, \dots, A_n are *pairwise intersecting* if $A_i \cap A_j$ is nonempty for all $i, j \in [n]$.

Definition 2.6. A set $I \subseteq \mathbb{R}$ is *convex* if $a, b \in I$ implies $[a, b] \subseteq I$.

Example 2.7 (Helly's theorem for \mathbb{R}^1). Let I_1, \dots, I_n be convex and pairwise intersecting subsets of \mathbb{R} . Then $I_1 \cap I_2 \cap \dots \cap I_n$ is nonempty.

Proof. We will prove this statement by induction. Let $P(n) :=$ “for any I_1, \dots, I_n as in the statement, $I_1 \cap I_2 \cap \dots \cap I_n$ is nonempty”. We will consider three base cases. The proposition $P(1)$ holds because $I_1 \cap I_1 = I_1$ is nonempty by the pairwise intersection property. Similarly, the proposition $P(2)$ holds because $I_1 \cap I_2$ is nonempty by the pairwise intersection property. The proposition $P(3)$ can be shown to be true by routine verification, which is not the point of this course; we will merely accept it as fact. Now assume that $P(k)$ holds for

some $k \in \mathbb{N}$; in other words, that $I_1 \cap I_2 \cap \dots \cap I_k$ is nonempty for any choice of I_1, \dots, I_k . Consider the intersection $I_1 \cap I_2 \cap \dots \cap I_k \cap I_{k+1}$, and let $J = I_k \cap I_{k+1}$. Then J is convex. Furthermore, $I_i \cap J = I_i \cap (I_k \cap I_{k+1})$ is nonempty by $P(3)$, so the sets $I_1, I_2, \dots, I_{k-1}, J$ are pairwise intersecting. Thus, as this is a collection of k sets, we have $I_1 \cap I_2 \cap \dots \cap I_{k-1} \cap J$ is nonempty by the inductive hypothesis $P(k)$. But $J = I_k \cap I_{k+1}$, so $I_1 \cap I_2 \cap \dots \cap I_{k-1} \cap J = I_1 \cap I_2 \cap \dots \cap I_{k+1}$, and thus $P(k) \implies P(k+1)$. Therefore, by the principle of mathematical induction, $I_1 \cap I_2 \cap \dots \cap I_n$ is nonempty for all $n \in \mathbb{N}$. \square

Remark 2.8. In some badly written proofs, the proposition P is not explicitly declared. This is definitely not a good example to follow, particularly since many inductive arguments turn out to be quite subtle.

3 Strong Induction and the Well-Ordering Principle

3.1 Strong Induction

Remark 3.1. The principle of mathematical induction is an incredibly powerful tool, but we need not “restrict” ourselves by assuming only $P(k)$ when trying to prove $P(k+1)$ for a proposition P . The following principle allows us to metaphorically free our hands from being tied behind our backs.

Definition 3.2. Let P be a proposition defined for all natural numbers. The *principle of strong mathematical induction* states that if $P(1) \wedge P(2) \wedge \dots \wedge P(k-1) \implies P(k)$ for all $k \in \mathbb{N}$, then $P(n)$ is true for all $n \in \mathbb{N}$.

Remark 3.3. In the context of strong mathematical induction, the *inductive hypothesis* refers to assuming $P(1) \wedge P(2) \wedge \dots \wedge P(k-1)$ holds for some $k \in \mathbb{N}$. Note that there is no need for the base case in the strong induction: it is already included since for $k=1$ the inductive hypothesis vanishes (becomes vacuously true).

Example 3.4 (Existence in fundamental theorem of arithmetic). *Let n be a natural number. Then n has a prime decomposition of $n = p_1^{d_1} p_2^{d_2} \dots p_u^{d_u}$ for some primes p_1, \dots, p_u and natural numbers d_1, \dots, d_u .*

Proof. We will prove this statement by strong mathematical induction. Let P be the proposition defined by $P(n) := n = p_1^{d_1} p_2^{d_2} \dots p_u^{d_u}$ for some primes p_1, \dots, p_u and natural numbers d_1, \dots, d_u , and assume that $P(1) \wedge P(2) \wedge \dots \wedge P(k-1)$ holds for some $k \in \mathbb{N}$. We will consider two cases.

► **Case I.** k is prime.

Then k is its own prime decomposition, so $P(k)$ holds (and we need not even use the inductive hypothesis).

► **Case II.** k is not prime.

Then $k = \ell m$ for some $\ell, m \in \mathbb{N}$ with $1 < \ell, m < k$. By our inductive hypothesis, we have $\ell = q_1^{e_1} q_2^{e_2} \dots q_v^{e_v}$ for some primes q_1, \dots, q_v and natural numbers e_1, \dots, e_v and $m = r_1^{f_1} r_2^{f_2} \dots r_w^{f_w}$ for some primes r_1, \dots, r_w and natural numbers f_1, \dots, f_w . Thus, k has a prime decomposition of $k = q_1^{e_1} q_2^{e_2} \dots q_v^{e_v} r_1^{f_1} r_2^{f_2} \dots r_w^{f_w}$, so $P(k)$ holds.

In either case, we have $P(1) \wedge P(2) \wedge \dots \wedge P(k-1) \implies P(k)$, so by the principle of mathematical induction, every natural number n admits a prime decomposition ($P(n)$ holds for all $n \in \mathbb{N}$). \square

Remark 3.5. Note that the above theorem is only one, and easy, direction of the fundamental theorem of arithmetic because the latter requires that the prime decomposition be *unique* (we will prove this later).

3.2 The Well-Ordering Principle

Definition 3.6. Let S be a set equipped with a (total) ordering \leq (orderings will be discussed in more detail later). The *least element* of S is the element $a \in S$ such that $a \leq x$ for all $x \in S$, if such an element exists. Similarly, the *greatest element* of S is the element $b \in S$ such that $x \leq b$ for all $x \in S$, if such an element exists.

Definition 3.7 (Well-ordering principle). The *well-ordering principle* states that every nonempty set of natural numbers contains a least element.

Remark 3.8. Although strong mathematical induction may appear to be strictly more powerful than ordinary induction, they are, in fact, equivalent. Furthermore, they are both equivalent to the well-ordering principle. Thus, in choosing an axiomatic system for the natural numbers, only one of these principles needs to be adopted as an axiom; the other two may simply follow as theorems, as the following theorem demonstrates. In this theorem, the word “logically” is deliberately left imprecise (all true statements are pairwise equivalent after all...); it could be thought of as something like “plainly”.

Theorem 3.9. *Ordinary induction, strong induction, and the well-ordering principle are logically equivalent.*

Proof. Suppose strong induction holds; we will show ordinary induction also holds. Then, in our inductive step, we may assume $P(1) \wedge P(2) \wedge \dots \wedge P(k-1)$ when trying to prove $P(k)$ for some proposition P and $k \in \mathbb{N}$, so to obtain ordinary induction, consider only $P(k-1)$ when trying to prove $P(k)$ (this is where the notion of “tying our hands behind our backs” comes from).

Now suppose ordinary induction holds; we will show the well-ordering principle also holds. Let X be a set of natural numbers, and define $X_n = X \cap [n]$. Consider the proposition P defined by $P(n) :=$ either X_n is empty or X_n contains a least element. We will use ordinary induction to prove P for all $n \in \mathbb{N}$. The base case, $P(1)$, is true because $X_1 = X \cap \{1\}$, so $X_1 = \{1\}$ or $X_1 = \emptyset$. In the former case, 1 is the least element of X_1 , and in the latter case X_1 is empty, so $P(1)$ holds. Now assume that $P(k)$ holds for some $k \in \mathbb{N}$. We will consider two cases.

► **Case I.** X_{k+1} is empty.

Then $P(k+1)$ is simply true.

► **Case II.** X_{k+1} is nonempty.

We will consider two further subcases.

► **Case i.** X_k is empty.

Then

$$\begin{aligned} X_{k+1} &= X \cap [k+1] \\ &= X \cap ([k] \cup \{k+1\}) \\ &= (X \cap [k]) \cup (X \cap \{k+1\}) \\ &= X_k \cup (X \cap \{k+1\}) \\ &= X \cap \{k+1\} \qquad (X_k \text{ is empty}), \end{aligned}$$

so $X_{k+1} = \{k+1\}$ or $X_{k+1} = \emptyset$. But we have supposed that X_{k+1} is nonempty, so $X_{k+1} = \{k+1\}$, and thus $k+1$ is the least element of X_{k+1} . Therefore, $P(k+1)$ is true.

► **Case ii.** X_k is nonempty.

Because X_k is nonempty, then X_k must have some least element j by $P(k)$. Then $j \in X_{k+1}$ because $X_k \subseteq X_{k+1}$. Furthermore, j must also be the least element of

X_{k+1} . This is because j is at most k , so therefore $j < k + 1$. Thus, $j \leq i$ for all $i \in X_k \cup \{k + 1\}$. But $X_{k+1} \subseteq X_k \cup \{k + 1\}$, so j is the least element of X_k . Therefore, $P(k + 1)$ is true.

In any case, we have $P(k) \implies P(k + 1)$, so by the principle of (ordinary) mathematical induction, $P(n)$ holds for all $n \in \mathbb{N}$.

Finally, suppose that the well-ordering principle holds; we will show strong induction also holds. Let Q be any proposition such that $Q(1) \wedge Q(2) \wedge \dots \wedge Q(k - 1)$ implies $Q(k)$ for all $k \in \mathbb{N}$. Let $X = \{n \in \mathbb{N} \mid Q(n) \text{ is false}\}$. Suppose for contradiction that X is nonempty. By the well-ordering principle, let y be the least element of X (so y is the minimal counterexample of Q). Then, as y is minimal, $Q(k)$ must be true for all $k < y$. In particular, we have $Q(1) \wedge Q(2) \wedge \dots \wedge Q(y - 1)$ holds, so by our assumption, $Q(y)$ also holds. But this is a contradiction, so X must be empty. Therefore, Q holds for all $n \in \mathbb{N}$. \square

Remark 3.10. When using the well-ordering principle, we typically suppose that a proposition P is false, then choose $n \in \mathbb{N}$ to be the minimal counterexample, and derive a contradiction from assuming n is minimal. This requires the well-ordering principle because we need to take the least element of the set $\{k \in \mathbb{N} \mid P(k) \text{ is false}\}$, as was done in the above proof.

The following example will investigate the so-called “finger-pointing” game, played by a finite set of people present in the same room. This game is played as follows: on the count of three, everyone in the room points to another person in the room with one finger.

Example 3.11. *There is always a cycle (loop) of people pointing in the finger-pointing game.* Suppose for contradiction that there is a configuration in which there is no cycle, and, by the well-ordering principle, let n be the smallest number of people for which the game admits such a configuration (the minimal counterexample to our statement).

Let A be any person. Remove A and have all the people that are pointing to A point to the person B to which A was pointing. Then there are $n - 1$ people in this configuration, so by $P(n - 1)$ (which must be true because n is the minimal counterexample to P and $n - 1 < n$), there must be a cycle in this configuration. If B is not in this cycle, we are done. Otherwise, we add A back into their original position, there must still be a cycle (but with a length increased by one, as A has been inserted into it). But this is a contradiction, as we have assumed that this configuration contains no cycles. Thus, our supposition that there exists a configuration in which there is no cycle must be false, so every configuration of the game must have a cycle.

Part IV

Graph Theory

21 Introduction to Graph Theory

21.1 Graphs

Definition 21.1. An (undirected) *graph* is a pair of sets (V, E) where V is a finite set of *vertices*, and E is a finite set of *edges*. For each $e \in E$, there are two associated vertices in V (not necessarily distinct) called *endpoints*.

Remark 21.2. We will deal almost exclusively with undirected graphs (graphs whose edges have no direction) in this course.

Remark 21.3. If G is a graph, we will use the notation $V(G)$ to refer to the set of vertices of G and $E(G)$ to refer to the set of edges of G .

Definition 21.4. Vertices v_1 and v_2 in a graph are *adjacent* if there is an edge in the graph whose endpoints are v_1 and v_2 .

Definition 21.5. A vertex v and an edge e in a graph are *incident* if one of the endpoints of e is v .

Remark 21.6. Often, the vertices of a graph with n vertices are represented by the integers $[n]$ and the edges are represented as pairs of integers (a, b) with $a, b \in [n]$.

Definition 21.7. A *simple graph* is a graph that does not contain duplicate edges (more than one edge between two vertices) or loops (an edge where each endpoint is the same vertex).

Remark 21.8. We will deal almost exclusively with simple graphs in this course.

Definition 21.9. A *complete graph* (also known as a *clique*) is a graph in which every pair of distinct vertices is adjacent. A complete graph with n vertices is denoted K_n .

Definition 21.10. A *complete bipartite graph* is a graph $G = (V, E)$ in which the set of vertices V is equal to the disjoint union of two sets A, B such that for every vertex $a \in A$ and $b \in B$, we have $(a, b) \in E$, and there are no other edges. A complete bipartite graph divided into sets of vertices of size m and n is denoted $K_{m,n}$.

Definition 21.11. A *cycle graph* is a connected (see Section 21.3 below) graph in which every vertex is the endpoint of two distinct edges. A cycle graph with n vertices is denoted C_n .

Remark 21.12. An *even cycle graph* is a cycle graph with an even number of vertices (C_2, C_4, C_6, \dots), and an *odd cycle graph* is a cycle graph with an odd number of vertices (C_1, C_3, C_5, \dots).

Definition 21.13. A *path graph* is a cycle graph with exactly one edge removed. A path graph with n vertices will be denoted by P_n .

21.2 The Handshaking Lemma

Definition 21.14. The *degree* of a vertex v in a graph G is the number of edges incident to v . This is denoted $\deg_G(v)$.

Remark 21.15. Equivalently, the degree of a vertex v in a (simple) graph G is the number of vertices to which v is adjacent.

Theorem 21.16 (Handshaking lemma). *Let $G = (V, E)$ be a graph with e edges. Then*

$$2e = \sum_{v \in V} \deg_G(v).$$

Proof. Let S be the set of tuples (e, v) where $e \in E$ and $v \in V$ such that e is incident to v . Then $|S| = 2e$ because each edge is incident to exactly two distinct vertices. On the other hand, $|S| = \sum_{v \in V} \deg_G(v)$ because each vertex $v \in V$ is incident to exactly $\deg_G(v)$ edges. Therefore, we have $2e = |S| = \sum_{v \in V} \deg_G(v)$. \square

21.3 Paths and Connectedness

Definition 21.17. A *path* in a graph G is a finite sequence of vertices (v_0, v_1, \dots, v_n) with $v_i \in V(G)$ for all $i \in \{0, 1, \dots, n\}$ such that $v_i \neq v_{i+1}$ and v_i is adjacent to v_{i+1} for all $i \in \{0, 1, \dots, n\}$ and no edge is repeated.

Definition 21.18. The length of a path (v_0, v_1, \dots, v_n) is n , the number of edges in the path.

Definition 21.19. A vertex s is *connected* to a vertex t if there exists a path from s to t , denoted $s \approx t$. This is an equivalence relation.

Remark 21.20. If a vertex s is connected to a vertex t by a path, we usually call s the *source* and t the *target* of this path.

Definition 21.21. The *connected components* of a graph G are a partition of G into subgraphs in which each vertex in a given subgraph is connected to every other vertex in the subgraph and no others.

Definition 21.22. A graph is *connected* if it has exactly one connected component.

Definition 21.23. The *distance* between two vertices u and v in a connected graph G is the minimum length of a path between vertices u and v . This is denoted $d(u, v)$.

Remark 21.24. If G is a connected graph, then (G, d) forms a metric space.

21.4 Graph Properties

Definition 21.25. The *minimum degree* of a graph G with vertices V is $\min_{v \in V} \deg_G(v)$. This is denoted $\delta(G)$.

Definition 21.26. The *maximum degree* of a graph G with vertices V is $\max_{v \in V} \deg_G(v)$. This is denoted $\Delta(G)$.

Definition 21.27. The *diameter* of a connected graph G is the maximum distance between two vertices in G . This is denoted $\text{diam}(G)$.

Theorem 21.28 (Moore bound). *Let G be a graph with n vertices. Then*

$$n \leq 1 + \Delta(G) \cdot \sum_{i=0}^{\text{diam}(G)-1} (\Delta(G) - 1)^i.$$

Definition 21.29. A graph G is a *Moore graph* if it has precisely

$$1 + \Delta(G) \cdot \sum_{i=0}^{\text{diam}(G)-1} (\Delta(G) - 1)^i$$

vertices (i.e., the Moore bound is an equality).

Definition 21.30. The *Petersen graph* is the following graph:

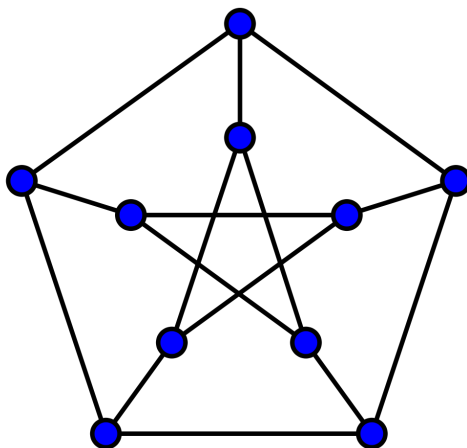


Image source: https://commons.wikimedia.org/wiki/File:Petersen1_tiny.svg.

Remark 21.31. The Petersen graph plays an important role in graph theory in that it serves as a counterexample to *many* conjectures that “look” true. When trying to come up with a counterexample to a conjecture, the Petersen graph should be one of the first candidates to inspect.

Theorem 21.32. *The Petersen graph is a Moore graph.*

Theorem 21.33. *The only possible Moore graphs with $\text{diam}(G) = 2$ must have $\delta(G) = 2, 3, 7, 57$.*

Remark 21.34. Moore graphs have been found with maximum degrees of 2, 3, and 7, but it is an open problem if there exists a Moore graph of degree 57. Note that the Petersen graph is a Moore graph of degree 3.

22 Exploring Graphs

22.1 Spectral Graph Theory

Remark 22.1. *Spectral graph theory* is a branch of mathematics that uses the tools of linear algebra to solve problems in graph theory, as we will see in the following results.

Question 22.2. Let G be a graph containing vertices u and v . How many paths from u to v are there of a given length ℓ ?

Definition 22.3. The *adjacency matrix* of a graph G with vertices $[n]$ is the n by n matrix denoted A_G with the following values:

$$(A_G)_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

Remark 22.4. The adjacency matrix of an undirected graph G is symmetric because $(u, v) \in E(G)$ if and only if $(v, u) \in E(G)$ for all vertices $u, v \in V(G)$.

Example 22.5. The adjacency matrix of the cycle graph C_5 with 5 vertices is

$$A_{C_5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Theorem 22.6. Let G be a graph with vertices $V = [n]$, and let $1 \leq i, j \leq n$. Then the number of paths from i to j of a given length ℓ is $(A_G^\ell)_{ij}$.

Proof. We will prove this statement by induction on ℓ . The base case of $\ell = 1$ follows directly from the definition of A_G . Now assume that the number of paths from i to j of length m is $(A_G^m)_{ij}$ for some $m \in \mathbb{N}$. Then

$$\begin{aligned} (A_G^{m+1})_{ij} &= (A_G^m A_G)_{ij} \\ &= \sum_{k=1}^n (A_G^m)_{ik} (A_G)_{kj} \quad (\text{matrix multiplication}). \end{aligned}$$

Note that $(A_G^m)_{ik}$ is the number of paths from i to k of length m by the induction hypothesis, and $(A_G)_{kj}$ is the number of paths from k to j of length 1 by the base case. Therefore, summing over all vertices k , we get that $\sum_{k=1}^n (A_G^m)_{ik} (A_G)_{kj}$ is the number of paths from i to j of length $m + 1$, exactly as desired. Hence, by the principle of mathematical induction, the number of paths from i to j of a given length ℓ is $(A_G^\ell)_{ij}$ for all $\ell \in \mathbb{N}$. \square

Question 22.7. What happens as $\ell \rightarrow \infty$ in the above theorem?

Definition 22.8. A graph G is *regular* if there exists some $d \in \mathbb{N}_0$ such that $\deg_G(v) = d$ for all $v \in V(G)$.

Remark 22.9. Let G be a connected, regular graph containing vertices u and v . As $\ell \rightarrow \infty$, the number of paths from u to v of length ℓ tends to d^ℓ .

Question 22.10. What happens as $\ell \rightarrow \infty$ for connected, non-regular graphs?

Remark 22.11. Let G be a connected, non-regular graph. Then, since A_G is symmetric, it can be diagonalized as

$$U^{-1}A_GU = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

for some (orthogonal but we do not care) matrix U , where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A_G . Then

$$U^{-1}A_G^\ell U = \begin{pmatrix} \lambda_1^\ell & 0 & \dots & 0 \\ 0 & \lambda_2^\ell & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^\ell \end{pmatrix}.$$

Now assume $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$. Then λ_1^ℓ dominates as $\ell \rightarrow \infty$, so the number of paths as between two vertices of length ℓ as $\ell \rightarrow \infty$ tends to λ_1^ℓ up to a lower order term.

22.2 Graph Relationships

Definition 22.12. Graphs G and H are *isomorphic* if there exists a bijection $\varphi : V(G) \rightarrow V(H)$ such that $(u, v) \in E(G)$ if and only if $(\varphi(u), \varphi(v)) \in E(H)$ for all $u, v \in V(G)$. This is denoted $G \approx H$.

Remark 22.13. Intuitively, graphs G and H are isomorphic if the vertices of G can be matched with the vertices of H in an edge-preserving way.

Definition 22.14. Let $G = (V, E)$ be a graph. Then a graph $G' = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$.

Definition 22.15. A graph G is a *supergraph* of a graph G' if G' is a subgraph of G .

Definition 22.16. Let G be a graph and let G' be a subgraph of G . Then G' is an *induced subgraph* of G if $(u, v) \in E(G')$ if and only if $(u, v) \in E(G)$, for all $u, v \in V(G')$.

Remark 22.17. Intuitively, given a graph G , a graph G' is an induced subgraph of G if only the necessary edges (and no others) are removed after removing any number of vertices. In other words, two vertices in $V(G')$ are adjacent if and only if they are adjacent in G (in a general subgraph of G , two adjacent vertices in G need not be adjacent in G').

Definition 22.18. Let G be a graph and let G' be a subgraph of G . Then G' is a *spanning subgraph* of G if $V(G') = V(G)$.

Remark 22.19. Intuitively, only edges (not vertices) may be removed from a graph to form a spanning subgraph.

Definition 22.20. Let $G = (V, E)$ be a graph. Then the complement of G is the graph $\bar{G} = (V, E')$, where $e \in E'$ if and only if $e \notin E$.

Remark 22.21. The complement of a graph G has the same vertices as G , but two vertices are adjacent in the complement of G if and only if they are *not* adjacent in G .

23 Some Graph Theorems

23.1 Extremal Graph Theory

Remark 23.1. *Extremal graph theory* is a branch of graph theory that studies how big (or small) particular graph quantities can be subject to other properties.

Question 23.2. What is the maximal number of edges in a graph with n vertices that does not contain any triangles?

Theorem 23.3 (Mantel's). *The maximal number of edges in a graph with n vertices that does not contain any triangles is $\lfloor n^2/4 \rfloor$.*

Proof. We will prove this theorem by strong induction on n . If a graph has one vertex, then it must have $0 = \lfloor 1^2/4 \rfloor$ edges; similarly, if a graph has two vertices, then it has a maximum of $1 = \lfloor 2^2/4 \rfloor$ edges. Therefore, the cases in which $n = 1, 2$ hold.

Now assume that the theorem holds for all $n \in \mathbb{N}$ such that $n \leq k$ for some $k \in \mathbb{N}$ and suppose that G is a graph with $k + 1$ vertices. Let u and v be adjacent vertices in G . Then every vertex in G must be adjacent to *at most* one of u and v , because otherwise G would contain a triangle. Therefore, $\deg_G(u) + \deg_G(v) \leq k + 1$.

Finally, let H be the subgraph of G induced by removing u and v . Then $|E(G)| = |E(H)| + d(u) + d(v) - 1$ (one is subtracted because u and v are adjacent, so $d(u) + d(v)$ double-counts the edge between u and v). Furthermore, H has $k - 1$ vertices, so by our inductive hypothesis, H has at most $\lfloor (k - 1)^2/4 \rfloor$ edges. Therefore,

$$\begin{aligned} |E(G)| &= |E(H)| + d(u) + d(v) - 1 \\ &\leq \lfloor (k - 1)^2/4 \rfloor + (k + 1) - 1 \\ &= \lfloor (k - 1)^2/4 \rfloor + k \\ &= \lfloor (k + 1)^2/4 \rfloor. \end{aligned}$$

Thus, by the principle of strong mathematical induction, the maximal number of edges in a graph with n vertices that does not contain a triangle is $\lfloor n^2/4 \rfloor$. holds for all $n \in \mathbb{N}$. \square

23.2 Bipartite Graphs

Definition 23.4. A *circuit* is a path with an additional edge between the source and the target vertices.

Remark 23.5. Every cycle is a circuit but not necessarily vice versa. An *even circuit* is a circuit with an even number of edges, and an *odd circuit* is a circuit with an odd number of edges.

Definition 23.6. A graph G is *bipartite* if its set of vertices can be partitioned into two disjoint subsets A, B such that for every edge $e \in E$, one endpoint of e is in A and the other is in B .

Remark 23.7. Complete bipartite graphs (the graphs $K_{m,n}$ for any $m, n \in \mathbb{N}$) are bipartite.

Lemma 23.8. *If G is a graph that contains no odd cycles, then G contains no odd circuits.*

Proof. Let G be a graph that contains no odd cycles, and suppose for contradiction that G has an odd circuit. Let C be the odd circuit in G with minimal length (well-ordering principle!), and let ℓ be the length of C . Note that there must be a repeated vertex v in the circuit C , or else C would be an odd cycle, which is contrary to the specification of G . Split C into two smaller circuits by the vertex v of length ℓ_1 and ℓ_2 (where $\ell_1, \ell_2 < \ell$). Then the combined length of the two new circuits is equal to the length of the original circuit, so $\ell_1 + \ell_2 = \ell$. But ℓ is odd, so one of ℓ_1 and ℓ_2 must be odd. However, is a contradiction

because $\ell_1, \ell_2 < \ell$, yet we have supposed that ℓ is the minimal length of an odd circuit in G . Thus, G must contain no odd circuits. \square

Theorem 23.9. *A (simple) graph G is bipartite if and only if G does not contain any odd cycles.*

Proof. Let G be a simple graph. Suppose G contains an odd cycle. Then G is not bipartite because odd cycles are not bipartite. Conversely, suppose that G contains no odd cycles. Let V_1 and V_2 be sets, which will be populated below. First consider any single connected component G' of G , and let u be any vertex in $V(G')$. Consider any vertex $v \in V(G')$. Then every path from u to v will have the same parity (oddness or evenness) because if there were two paths P_1, P_2 from u to v such that P_1 was odd and P_2 was even, then P_1 and P_2 would form an odd circuit around u , which would contradict **Lemma 23.8**. Therefore, every vertex $v \in V(G')$ can be classified into two disjoint sets: V_1 if the parity of the path length from u to v is even, and V_2 if it is odd. Then each vertex $a \in V_1$ cannot be adjacent to any other vertex $b \in V_1$ because then there would be an odd cycle around a, b , and u , which is impossible by **Lemma 23.8**; similar logic follows for V_2 . We may generalize this argument to every connected component of G , and because no connected component has an edge that connects it to a different connected component, V_1 and V_2 form a bipartite partition of the full graph G . \square

24 More Graph Properties and Trees

24.1 Cliques, Coloring, and Independence

Definition 24.1. The *clique number* of a graph G is the largest integer r such that G contains K_r . This is denoted $\omega(G)$.

Definition 24.2. A graph G is triangle-free if $\omega(G) \leq 2$. In other words, a graph is triangle-free if it does not contain K_3 (the triangle graph).

Definition 24.3. A (proper) *coloring* of a graph G is a function $f : V(G) \rightarrow \{1, 2, \dots, C\}$ for some $C \in \mathbb{N}$ such that if $(u, v) \in E$, then $f(u) \neq f(v)$.

Remark 24.4. A coloring of a graph is a categorization of vertices into distinct “colors” such that no two adjacent vertices have the same color.

Definition 24.5. The *chromatic number* of a graph G is the minimum size of the image of a coloring of G . This is denoted $\chi(G)$. The chromatic number represents the minimum number of colors needed to properly color a graph.

Theorem 24.6. A graph G is bipartite if and only if $\chi(G) \leq 2$.

Proof. Suppose G is bipartite with a partition of vertices into sets V_1 and V_2 . Then simply assign all the vertices in V_1 one color and all the vertices in V_2 a second color. This is a proper coloring of G , so $\chi(G) \leq 2$.

Conversely, suppose that $\chi(G) \leq 2$. Then let $f : V(G) \rightarrow \{1, 2\}$ be a coloring of G . Let $V_1 = f^{-1}(\{1\})$ and $V_2 = f^{-1}(\{2\})$. Then every vertex in V_1 has the same color, so no two vertices may be adjacent. The same goes for V_2 . Therefore, V_1 and V_2 form a bipartite partition of G , so G is bipartite. \square

Theorem 24.7. Let G be a graph. Then $\omega(G) \leq \chi(G)$.

Proof. Let $r = \omega(G)$. Then G contains K_r , which requires r colors to properly color because each vertex of K_r is connected to every other vertex of K_r . Thus, $\chi(G) \geq r = \omega(G)$. \square

Theorem 24.8 (Greedy algorithm). Let G be a graph. Then $\chi(G) \leq \Delta(G) + 1$.

Proof. Let $v \in V(G)$. Then v is adjacent to at most $\Delta(G)$ vertices, so simply (“greedily”) color every single vertex adjacent to v a different color. Finally, color v itself yet a different color (for a total of $\Delta(G) + 1$ colors). Branch out from v and consecutively color each new vertex a color different from the colors of its already-colored neighbours until the entire graph is colored (repeat as necessary for the different connected components of G). \square

Definition 24.9. A set of vertices is *independent* (or *stable*) if no two vertices in the set are adjacent.

Definition 24.10. Let G be a graph with complement \bar{G} . Then the *independence number* of G is equal to $\omega(\bar{G})$. This is denoted $\alpha(G)$. The independence number of a graph G is the size of the largest set of independent vertices in G .

Theorem 24.11. Let G be a graph with n vertices. Then $\chi(G)\alpha(G) \geq n$.

Proof. Let $f : V(G) \rightarrow \{1, 2, \dots, \chi(G)\}$ be a coloring of G . Then $V(G) = \bigcup_{c=1}^{\chi(G)} f^{-1}(c)$, so $|V(G)| = \left| \bigcup_{c=1}^{\chi(G)} f^{-1}(c) \right|$. However, we have $\left| \bigcup_{c=1}^{\chi(G)} f^{-1}(c) \right| = \sum_{c=1}^{\chi(G)} |f^{-1}(c)|$ because each set of colored vertices is disjoint from every other set, so $|V(G)| = \sum_{c=1}^{\chi(G)} |f^{-1}(c)|$.

Finally, note that each set of colored vertices is independent by the definition of a coloring, so $|f^{-1}(c)| \leq \alpha(G)$ for all $c \in \{1, 2, \dots, \chi(G)\}$. Hence,

$$\begin{aligned}
 n &= |V(G)| \\
 &= \sum_{c=1}^{\chi(G)} |f^{-1}(c)| \\
 &\leq \sum_{c=1}^{\chi(G)} \alpha(G) \\
 &= \chi(G)\alpha(G). \quad \square
 \end{aligned}$$

Corollary 24.11.1. *Let G be a graph with n vertices. Then $\chi(G) \geq \frac{n}{\alpha(G)}$.*

Proof. By **Theorem 24.11**, $\chi(G)\alpha(G) \geq n$, so $\chi(G) \geq \frac{n}{\alpha(G)}$. \square

Remark 24.12. **Corollary 24.11.1** is useful in that it provides a lower bound for the value of $\chi(G)$, which may be otherwise hard to compute.

24.2 Trees

Definition 24.13. A graph is *acyclic* if it does not contain any cycles.

Definition 24.14. A *tree* is a connected, acyclic graph.

Definition 24.15. A graph G is *minimally connected* if G is connected but any spanning subgraph of G with strictly fewer edges is not connected.

Definition 24.16. A graph G is *maximally acyclic* if G is acyclic but any supergraph of G with the same vertices but strictly more edges is cyclic.

Theorem 24.17. *Let G be a simple graph. Then the following are pairwise equivalent:*

- (a) G is a tree.
- (b) G is connected and has $V(G) - 1$ edges.
- (c) G is acyclic and has $V(G) - 1$ edges.
- (d) For every pair of distinct vertices $s, t \in V(G)$, there exists a unique simple path from s to t .

(e) G is minimally connected.

(f) G is maximally acyclic.

Proof of (a) \iff (f). Suppose G is a tree and let H be a supergraph of G with the same number of vertices but strictly more edges, so then there exists some $(u, v) \in E(H) \setminus E(G)$. Then, because G is a tree, G is connected, so there exists a path P in G from u to v . Then the edge (u, v) in $E(H)$ forms a cycle with the path P , so H is cyclic. Thus, G is maximally acyclic.

Conversely, suppose that G is maximally acyclic. Suppose for contradiction that G is not connected. Then there exist vertices $u, v \in G$ such that there is no path from u to v in G . Let H be the supergraph of G defined by $H = (V(G), E(G) \cup \{(u, v)\})$. Then H must be acyclic because there can be no cycle created with the single vertex added to G , as there is no path from u to v in H that does not use the edge (u, v) . But this is a contradiction because G is maximally acyclic. Therefore, G must be connected. Thus, because G is acyclic and connected, G is a tree. \square

25 Planar Graphs

25.1 Planarity

Definition 25.1. A *planar embedding* of a graph G is an embedding of G into a (two-dimensional) plane such that the edges of G intersect only at the vertices of G .

Definition 25.2. A graph is *planar* if it admits at least one planar embedding.

Remark 25.3. The graphs K_5 and $K_{3,3}$ are both important nonplanar graphs, as will be seen later in this section.

Definition 25.4. A *subdivision* of a graph G is a graph such that any number of edges in G are replaced with path graphs of any size.

Definition 25.5. A *minor* of a graph G is a graph that can be formed by removing any number of edges and vertices in G and “contracting” any number of edges in G into single vertices.

Theorem 25.6. *If a graph G is planar, then all subgraphs of G , subdivisions of G , and minors of G are also planar. For subdivisions, the converse is also true.*

Remark 25.7 (Four color theorem). If G is a planar graph, then $\chi(G) \leq 4$. This is very hard to prove, and the only known proof requires the use of a computer!

25.2 Euler's Formula

Definition 25.8. A *region* of a planar embedding of a graph G is the area enclosed by any set of vertices and edges, including the infinitely-large exterior of a graph.

Definition 25.9. A *bridge* in a graph G is an edge whose removal increases the number of connected components.

Theorem 25.10 (Euler's formula). *Let G be a planar graph. Let c be the number of connected components of G , let v be the number of vertices in G , let e be the number of edges in G , and let r be the number of regions in a planar embedding of G . Then*

$$c - v + e - r = -1.$$

Proof. We will prove this theorem by induction on e . If $e = 0$, then G must have exactly v connected components and one region, so $c = v$ and $r = 1$. Therefore, $c - v + e - r = v - v + 0 - 1 = -1$, so the theorem holds for the base case of $e = 0$. Now assume that the theorem holds for some $e = k \in \mathbb{N}_0$ and suppose that G has $k + 1$ edges. Let $\beta \in E(G)$ and consider the subgraph $H = (V(G), E(G) \setminus \{\beta\})$ of G . Let c_H be the number of connected components of H and let r_H be the number of regions in a planar embedding of H . Note that H has precisely k edges, so by the inductive hypothesis, $c_H - v + k - r_H = -1$. We will now consider two cases.

► **Case I.** The removed edge β is a bridge in H .

Then H has one more connected component than G and the same number of regions, so $c_H = c + 1$ and $r_H = r$. Therefore, $(c + 1) - v + k - r = -1$, so $c - v + (k + 1) - r = -1$.

► **Case II.** The removed edge β is not a bridge in H .

Then H has the same number of connected components as G but one less region, so $c_H = c$ and $r_H = r + 1$. Therefore, $c - v + k - (r + 1) = -1$, so $c - v + (k + 1) - r = -1$.

In either case we have $c - v + (k + 1) - r = -1$, so by the principle of mathematical induction, the theorem holds for all $e \in \mathbb{N}_0$. \square

Corollary 25.10.1. *Let G be a connected planar graph. Let v be the number of vertices in G , let e be the number of edges in G , and let r be the number of regions in a planar embedding of G . Then*

$$v - e + r = 2.$$

Proof. Note that G is connected, so it has a single connected component. Therefore, by **Theorem 25.10**, we have $1 - v + e - r = -1$, so $v - e + r = 2$. \square

Remark 25.11. The quantity $v - e + r$ is called the *Euler characteristic*. The quantity v is said to have dimension 0, the quantity e is said to have dimension 1, and the quantity r is said to have dimension 2. In general, the Euler characteristic is the alternating sum of quantities with n dimensions, and it can be computed for an arbitrary surface. The result will be $2g - 2$, where g is the *genus* of the surface, and it again does not depend on the choice of the connected graph G . More generally, quantities analogous to the Euler characteristic are extremely important in virtually all branches of modern mathematics.

25.3 Dual Graphs

Definition 25.12. The *dual graph* G^* of a graph G with a planar embedding is created by letting each region in the planar embedding of G be a vertex in G^* which is adjacent to the vertices in G^* that represent adjacent regions in the planar embedding of G .

Remark 25.13. Note that G^* depends both on G *as well as* its planar embedding, not just G as an abstract graph. Hence, a planar graph G may have many dual graphs depending on its planar embedding. However, the number of edges in all of them will be the same and equal to the number of edges in G . Note also that if G is a graph with a planar embedding G^* , then the dual of G^* is G .

Theorem 25.14. *If G is a simple graph with at least one region, then $\delta(G^*) \geq 3$.*

Proof. Each region in G must be enclosed by at least three edges, so each vertex in G^* must have degree at least 3. Thus, $\delta(G^*) \geq 3$. \square

25.4 The Nonplanarity of K_5 and $K_{3,3}$

Theorem 25.15. *If G is a connected planar graph with at least one face that has v vertices and e edges, then $e \leq 3v - 6$.*

Proof. Let R be the set of regions in a planar embedding of G . Then

$$\begin{aligned} 2e &= \sum_{v \in V(G^*)} \deg_{G^*}(v) && \text{(Theorem 21.16) applied to } G^* \\ &\geq \sum_{v \in V(G^*)} \delta(G^*) \end{aligned}$$

$$\begin{aligned}
&\geq \sum_{v \in V(G^*)} 3 && \text{(Theorem 25.14)} \\
&= 3 |V(G^*)| \\
&= 3 |R| \\
&= 3(e - v + 2) && \text{(Euler's formula)} \\
&= 3e - 3v + 6,
\end{aligned}$$

so $e \leq 3v - 6$. □

Corollary 25.15.1. *Every planar graph has at least one vertex with degree at most 5.*

Proof. Let G be a graph with a planar embedding with v vertices and e edges. Suppose for contradiction that every vertex in G has degree at least six. Then, by the handshaking lemma, we have $2e \geq 6v$. However, by **Theorem 25.15**, we have $2e \leq 6v - 12$, a contradiction. □

Corollary 25.15.2. *If G is a planar graph, then $\chi(G) \leq 6$.*

Corollary 25.15.3. *The complete graph K_5 is not planar.*

Proof. K_5 has five vertices and ten edges and $10 > 3 \cdot 5 - 6 = 9$, so by **Theorem 25.15**, K_5 is not planar. □

Theorem 25.16. *If G is a bipartite planar graph with v vertices and e edges, then $e \leq 2v - 4$.*

Proof. (sketch) Since G is triangle-free, **Theorem 25.14** improves to $\delta(G^*) \geq 4$. Now do the same computation as in the proof of **Theorem 25.15**. □

Corollary 25.16.1. *The complete bipartite graph $K_{3,3}$ is not planar.*

Proof. $K_{3,3}$ has six vertices and nine edges and $9 > 2 \cdot 6 - 4 = 8$, so by **Theorem 25.16**, $K_{3,3}$ is not planar. □

Theorem 25.17 (Kuratowski's). *A graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.*