# Quantum Computing

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You may work together on solving homework problems, but put all the names of your collaborators clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Shopping for solutions on the Internet (or asking ChatGPT and such) is strongly discouraged but if you do it anyway, you must completely understand the proof, explain it in your own words and include the URL. On the contrary, shopping for useful facts is encouraged.

Prove all your answers with ''reasonable'' degree of rigor.

PDF file **prepared from a TeX source** is the preferred format. In that case you will get back your feedback in a neat annotated form.

## Homework 2, due February 21

1. Someone is trying to compute the order of $a \in \mathbb{Z}_N^*$ using the operator $V_a |x\rangle = |ax + 1 \bmod N\rangle$ instead of $U_a |x\rangle = |ax \bmod N\rangle$. Under the assumption $(a-1) \in \mathbb{Z}_N^*$, will this effort be successful? If you answer "yes", specifically address the following potential issues.

   (a) How to compute $V_a^{2^h}$ efficiently?

   (b) What state should be used for initialization during the second stage of the algorithm (in Shor's original algorithm it is $|1\rangle$).

2. Consider the following generalization of Deutsch-Jozsa:

$$F(X_1, \ldots, X_N) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } X = 0^N \text{ or } 1^N \\ 1 & \sum_{x \in [N]} X_x \in [\ell, N - \ell] \\ \text{undefined} & \text{otherwise} \end{cases}$$

(the original DS corresponds to the case $\ell = N/2$).

Prove that $Q_2(F) = \Theta\left(\sqrt{\frac{N}{\ell}}\right)$.

`Note. ‘‘Θ’’ means ‘‘equal up to an absolute multiplicative constant.’’`

3. Let us call a Boolean string $A_1, \ldots, A_N$ *mid-Western* if there exists at least one index $c$ such that $A_c = A_{c+1} = 1$ while $A_m = 0$ for any other $m$ with $|c - m| \leq \sqrt{N}$. Let $MW_N(X_1, \ldots, X_N)$ be the characteristic function of the set of all mid-Western strings.

   Determine its sensitivity and block-sensitivity up to "Θ".

4. (a) Prove that any total function $F(X_1, \ldots, X_N)$ with $\widetilde{\deg}(F) \leq 1$ essentially depends on at most five variables.

   (b) Give an example of such a function that does depend on five variables.

   `Hint. It might be helpful to change the` $\{0,1\}$ `notation to the` $\{\pm 1\}$ `notation.`

5. $\mathrm{CLIQUE}_{3,N}$ is the Boolean function in $\binom{N}{2}$ variables encoding an $N$-vertex graph that outputs one if and only if this graph contains a triangle. Prove that

$$\Omega(N) \leq Q_2(\mathrm{CLIQUE}_{3,N}) \leq O(N^{3/2}).$$

   `Note. Narrowing this gap is a long-standing open problem for which known methods seem to fail completely.`