Quantum Computing

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You may work together on solving homework problems, but put all the names of your collaborators clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Shopping for solutions on the Internet (or asking ChatGPT and such) is strongly discouraged but if you do it anyway, you must completely understand the proof, explain it in your own words and include the URL. On the contrary, shopping for useful facts is encouraged.

Prove all your answers with ''reasonable'' degree of rigor.

PDF file **prepared from a TeX source** is the preferred format. In that case you will get back your feedback in a neat annotated form.

Homework 3, due March 7

1. Let $1 \leq \ell \leq N/2$ and consider the **total** function $F(X_1, \ldots, X_N)$ given by $F(X_1, \ldots, X_N) = 1$ iff $\sum_{x \in [N]} X_x \in [\ell, N - \ell]$.

   Prove that $Q_2(F) \geq \Omega(\sqrt{N})$ regardless of the value of ℓ.

2. We proved in class that $QC_2(\mathrm{IP2}_n) \geq \Omega(n)$, where QC_2 is the quantum communication complexity and $\mathrm{IP2}_n(x, y) \stackrel{\mathrm{def}}{=} \bigoplus_{i=1}^n (x_i \wedge y_i)$. I also mentioned my result that $QC_2(\mathrm{DISJ}_n) \geq \Omega(\sqrt{n})$, where $\mathrm{DISJ}_n(x, y) \stackrel{\mathrm{def}}{=} \bigvee_{i=1}^n (x_i \wedge y_i)$.

   Which of these bounds will hold for the function $\bigvee_{i=1}^n (x_i \oplus y_i)$?

3. For any two pure 1-qubit states $|\phi\rangle, |\psi\rangle$ we now have two notions of "angle" between them: the "physical" angle $A(|\phi\rangle, |\psi\rangle) \stackrel{\mathrm{def}}{=} \arccos(|\langle\phi|\psi\rangle|) \in [0, \pi/2]$ and the "Bloch" angle $B(|\phi\rangle, |\psi\rangle) \stackrel{\mathrm{def}}{=} \arccos(\langle S_{|\phi\rangle}, S_{|\psi\rangle}\rangle) \in$

$[0, \pi]$, where $S_{|\phi\rangle}, S_{|\psi\rangle} \in \mathbb{R}^3$ are their Bloch vectors. Since unitary operators act conformally (angle-preserving) in both spaces, by [3, Theorem 4.2.1] there must be a formula expressing the Bloch angle in terms of the physical angle. Find it, as a closed-form expression, and prove it.

4. Recall that a (pure) two-qubit state is *non-entangled* if it has the form $|\phi\rangle \otimes |\psi\rangle$, where $|\phi\rangle, |\psi\rangle$ are 1-qubit states.

   Prove that there does not exist a probability distribution on pure non-entangled states that has the same density matrix as the EPR state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

5. Prove or disprove the following. Every superoperator T acting from an N-dimensional Hilbert space to itself always preserves the completely depolarized state $\frac{1}{N} I_N$.