Discrete Mathematics

Instructor: Alexander Razborov, University of Chicago
razborov@cs.uchicago.edu
Course Homepage: www.cs.uchicago.edu/~razborov/teaching/autumn09.html

Autumn Quarter, 2009

Prove all of your answers. Unless otherwise stated, you may use any method. The choice of the proof method will not affect your grade but if we get some particularly elegant and/or unexpected proofs, we can do them in the class.

If you work with others put their names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Homework is due at the beginning of class.

Homework 6, due November 18

1. Let the function $f : [n] \longrightarrow [n]$ be selected at random from the set of all functions from $[n] = \{1, 2, \ldots, n\}$ to itself. What is the expected number of those pairs $\langle i, j \rangle$ for which $i \leq j$ and $f(i) \leq f(j)$?

2. Find a graph G on 6 vertices with the degree sequence 4,4,2,2,1,1.

3. Let $a, b \geq 2$ be integers. Consider the graph $G = (V, E)$, where $V = \{0, 1, \ldots, a-1\}$ and $E = \{\, (x, x + b \mod a) \,|\, x \in V \,\}$. Prove that G is connected if and only if $gcd(a, b) = 1$.

4. Let $n \geq 4$. Prove that a graph on n vertices that contains *at most* one triangle, still may contain at most $\mathrm{ex}(n; K_3)$ edges, where

$$\mathrm{ex}(n; K_3) = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd.} \end{cases}$$

5. Prove that a graph is bipartite if and only if it does not contain any *induced* cycle C_r of odd length r.