Graph Theory

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Spring Quarter, 2012

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You are encouraged to work together on solving homework problems,
but please put their names clearly at the top of the assignment.
Everyone must turn in their own independently written solutions.
Homework is due at the beginning of class.
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Homework 4, due April May 2

1. Let G be an undirected simple graph and $u, v, w \in V(G)$ be three different vertices.

   (a) Prove[1] that the three integers $\bar{\lambda}(u,v), \bar{\lambda}(u,w), \bar{\lambda}(v,w)$ can not be pairwise different.

   (b) Does a similar statement hold for $\lambda(u,v), \lambda(u,w), \lambda(v,w)$ if we additionally assume that u, v, w are pairwise non-adjacent?

2. Prove that for any non-negative function f on the set of arcs $A(\Gamma)$ and any sets of nodes X, Y we have $f^+(X \cup Y) + f^+(X \cap Y) \leq f^+(X) + f^+(Y)$.

3. A cut $[S, \bar{S}]$ in an undirected simple graph is called *maximum* if it has maximal possible size among all cuts.

   (a) Prove that if the cut $[S, \bar{S}]$ is maximum then for every $v \in S$ we have[2] $|N(v) \cap \bar{S}| \geq |N(v) \cap S|$, and for every $v \in \bar{S}$ we have $|N(v) \cap \bar{S}| \leq |N(v) \cap S|$.

[1] Recall that $\bar{\lambda}(x,y)$ is the largest possible number of edge-disjoint paths from x to y.
[2] $N(v)$ is the set of all vertices adjacent to v.

(b) Prove that the converse to the above statement need not necessarily be true.

4. Find a maximum flow and a minimum cut in the network shown on Figure 1.

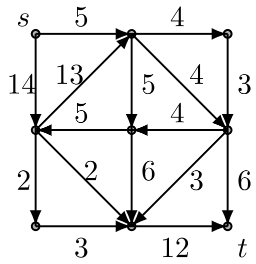

Figure 1: Just a network

5. Several people form n committees in such a way that every set S of the committees collectively has at least $\frac{|S|}{2}$ members. Prove that one can choose at least $(n/2)$ committees and assign one chair to each selected committee in such a way that no one serves as a chair for two different committees.